Complex Numbers

History

"The historical development of complex number" D.R.Green Mathematical Gazette June 1976 pp99-107.
In **N** we cannot solve $x + 2 = 1$
In **Z** we cannot solve $2x = 1$
In **Q** we cannot solve $x^2 = 2$
In **R** we cannot solve $x^2 + 1 = 0$
You have all done some work on complex numbers, and this introduction is in the spirit of the construction from **Z** to **Q**.

Definition

A complex number is an ordered pair (x, y) of real numbers, with addition and multiplication defined by

$$\begin{aligned} (x, y) + (x', y') &= (x + x', y + y') \\ (x, y).(x', y') &= (xx' - yy', xy' + yx') \end{aligned}$$

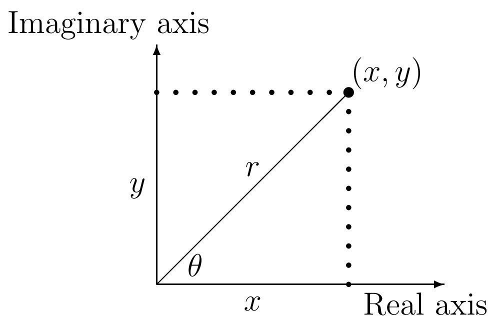
With these definitions the complex number system **C** has all the properties of a field. Now we have

$$\begin{aligned} (x, 0) + (x', 0) &= (x + x', 0) \\ (x, 0)(x', 0) &= (xx', 0) \end{aligned}$$

so there is a subsystem which behaves like **R**.
$(0, 1)(0, 1) = (-1, 0)$
$(x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0)(0, 1)$
We shall abbreviate $(x, 0)$ to x and $(0, 1)$ to i.
So we write $(x, y) = x + yi$
x is called the *real part* of the complex number.
y is called the *imaginary part* of the complex number.
Using this new symbolism we have:
$(x + yi) + (x' + y'i) = (x + x') + (y + y')i$
$(x + yi)(x' + y'i) = (xx' - yy') + (xy' + yx')i$

The Complex Plane

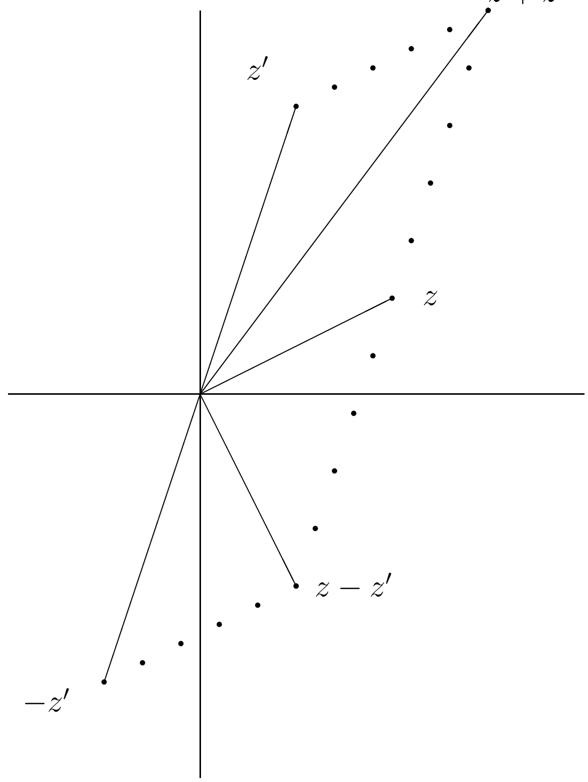
We can represent $x + yi$ as a point in the plane with coordinates (x, y).



If we write $z = x + yi$ then we have $x = r\cos\theta \qquad y = r\sin\theta$
So $z = r(\cos\theta + i\sin\theta)$ - Polar form of z.
r is called the modulus of z; $|z| = \sqrt{(x^2 + y^2)}$
θ is called the argument of z; $\arg z$ it satisfies $\tan\theta = \dfrac{y}{x}$
There are many values of θ satisfying $\tan\theta = \frac{y}{x}$. The value of θ is taken to satisfy $-\pi < \theta \leq \pi$ and this is called the principal argument of z.
So

$$\begin{aligned}
\arg(1+i) &= \frac{\pi}{4} \\
\arg i &= \frac{\pi}{2} \\
\arg -1 &= \pi \\
\arg(-1-i) &= -\frac{3\pi}{4}
\end{aligned}$$

Note that to say $\theta = \tan^{-1}\dfrac{y}{x}$ is not correct, for it does not distinguish $1 + i \ (x = y = 1)$ from $-1 - i \ (x = y = -1)$.
Addition in the complex plane is interpreted geometrically through the parallelogram law.



Triangle inequality $|z + z'| \leq |z| + |z'|$

Example

Prove from the triangle inequality that
$||z| - |z'|| \leq |z + z'|$

$|z| - |z'| = |(z + z') - z'| - |z'| \leq |z + z'| + |z'| - |z'| = |z + z'|$
Similarly $|z'| - |z| \leq |z' + z|$
Thus $||z| - |z'|| \leq |z + z'|$

Multiplication is best approached using the polar form.
Let $z = r(\cos\theta + i\sin\theta)$; $\qquad z' = r'(\cos\theta' + i\sin\theta')$
Multiplying it is easily verified that $zz' = rr'(\cos(\theta + \theta') + i\sin(\theta + \theta'))$
Thus we have $|zz'| = rr' = |z||z'|$
$\arg zz' = \arg z + \arg z' \ (\mathrm{mod} 2\pi)$

Exercise

Prove by induction that $|z^n| = |z|^n$
$\arg z^n = n \arg z \ (\mathrm{mod} 2\pi) \quad n\epsilon\ \mathbf{N}$
If $m = -n \quad n\epsilon\mathbf{N}$
Then $z^m z^n = 1$ So $|z^m||z^n| = 1$
i.e. $|z^m||z|^n = 1$
so $|z^m| = \dfrac{1}{|z|^n} = |z|^m$

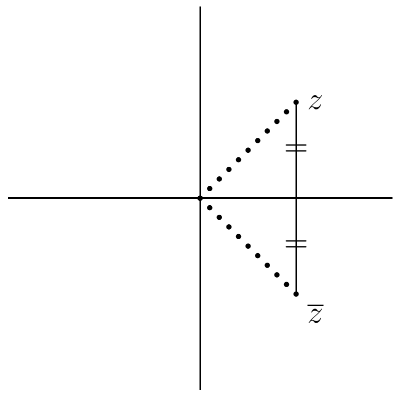
Exercise
Prove that if $m = -n \quad n \epsilon \mathbf{N}$
then $\arg z^m = m \arg z \bmod 2\pi$
The most important feature of the complex number system is that not only does $x^2 + 1 = 0$ have a solution in $\mathbf{C}$, but all polynomial equations have solutions in $\mathbf{C}$.
This fact was first given a complete proof by Gauss in 1799.

Fundamental Theorem of Algebra
Let $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n \quad a_i \epsilon \mathbf{C}$
Then the equation $p(z) = 0$ has a solution in $\mathbf{C}$
It follows that if c is a such solution then $p(z) = (z-c)(b_0+b_1z+...+b_{n-1}z^{n-1})$
Exercise
Try to prove this.
Corollary
$p(z)$ can be expressed as a product of n linear factors.
$p(z) = a_n(z - c_1)(z - c_2)...(z - c_n)$ where some of the c_i may be equal. Thus every polynomial has at most n roots in $\mathbf{C}$.
Proof by induction is left as an exercise.

Examples

$$\begin{array}{rcll} x^2+1 & & \text{irreducible} & \text{over } \mathbf{R} \\ x^2+1 & = & (x+i)(x-i) & \text{over } \mathbf{C} \\ x^3-x^2+2x-2 & = & (x-1)(x^2+2) & \text{over } \mathbf{R} \\ x^3-x^2+2x-2 & = & (x-1)(x+i\sqrt{2})(x-i\sqrt{2}) & \text{over } \mathbf{C} \end{array}$$

Complex conjugates
Let $z = x + iy$. Then we define the complex conjugate of z to be $\overline{z}$ or $z^* = x - iy$



Properties

i) $\overline{z+w} = \overline{z} + \overline{w}$ direct verification

ii) $\overline{zw} = \overline{z}\,\overline{w}$ direct verification

iii) $\overline{z^n} = (\overline{z})^n$ from ii) by induction

iv) If p is a polynomial $p(z) = a_0 + a_1 z + ... + a_n z^n$
$\overline{p(z)} = \overline{a}_0 + \overline{a}_1 \overline{z} + \overline{a}_2 \overline{z}^2 + ... + \overline{a}_n \overline{z}^n$

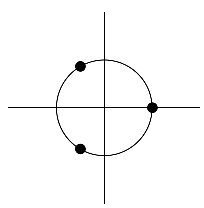
This result is used to prove that if z is a root of a polynomial with real coefficients then $\overline{z}$ is also a root. For in this case if $a_i \epsilon$ **R** then $\overline{a}_i = a_i$.
So $\overline{p(z)} = p(\overline{z})$
Thus if $p(z) = 0 \;\; p(\overline{z}) = 0$ also.
So for a real polynomial all the complex roots have corresponding conjugates.
Thus a real polynomial of odd degree must have at least one real root.

v) $z\overline{z} = (x+iy)(x-iy) = x^2 + y^2$
This is useful in such situations as
$$\frac{3+4i}{2-i} = \frac{(3+4i)(2+i)}{5} = \frac{2+11i}{5}$$

vi) $\dfrac{z+\overline{z}}{2} = \text{Re } z = x$
$\dfrac{z-\overline{z}}{2} = i \text{ Im } z = iy$

De Moivre's Theorem
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
Proof by induction left as an exercise.

The roots of unity
We consider the equation $z^n = 1$.
If $z = r(\cos\theta + i\sin\theta)$
$z^n = r^n(\cos n\theta + i\sin n\theta)$
So $r^n = 1$ which gives $r = 1$, and $\cos n\theta = 1$.
This gives $\theta = 0, \dfrac{2\pi}{n}, \dfrac{4\pi}{n}, ..., \dfrac{(2n-2)\pi}{n}$
Thus the solutions of $z^n = 1$ are
$\cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$ $\;k = 0, 1, ...n-1$ and these are all different.
For example take $n = 3$ the roots of $z^3 = 1$ are
$1,\ \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3},\ \cos 4\pi 3 + i\sin\dfrac{4\pi}{3}$
$1,\ \dfrac{-1+\sqrt{3}i}{2},\ \dfrac{-1-\sqrt{3}i}{2}$
They lie at the vertices of a regular triangle on the unit circle in the complex plane.



In general the n-th roots of unity lie at the vertices of a regular n-gon.

Exercise
Let the cube roots of unity be denoted by 1, w_1, w_2. Prove that $w_1^2 = w_2$ and $w_2^2 = w_1$.
Prove that w_1, w_1^2, w_1^3 are all different, and w_2, w_2^2, w_2^3 are all different, and each set is a permutation of 1, w_1, w_2. Investigate this situation for $n = 5, 8$ and then see if you can make any general statements for the n-th roots of unity.
Useful in this exercise might be:

Euler's formula

Assuming the series for

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

Replace x by ix in the first and separate the resulting series into real and imaginary parts to verify that

$e^{ix} = \cos x + i \sin x$

Then using $e^{-ix} = \cos x - i \sin x$

We obtain $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

These look like the formula for hyperbolic functions. Investigate this connection further.

Euler's formula enables us to deal with the roots of unity more concisely.

Since $\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} = e^{i\frac{2k\pi}{n}}$

we can obtain them as follows

$z^n = 1 = e^{2\pi i} = e^{4\pi i} = ... = e^{2k\pi i}$

so $z = e^{\frac{2k\pi i}{n}}$

The formula with DeMoivre's theorem is useful in summing series and evaluating integrals.