QUESTION

Consider the Black-Scholes equation for the continuous-time value of an option $V(S,t)$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where S is the price of the asset at time t, σ is the volatility and where the risk-free rate of return is r.

(a) Show that the equation, satisfied by the discounted option value $U(S,t)$ relative to the maturity date of the option T, where,

$$V(S,t) = \exp(-r(T-t))U(S,t),$$

is given by

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0 \tag{1}$$

(b) Hence show that under a change of variables to backwards time $\tau = T - t$ and log-prices $\xi = \log S$,

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}. \tag{2}$$

(c) Hence show that a further change of variables, which you should specify, can convert equation (2) into the diffusion equation,

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}, \quad W = W(x,\tau). \tag{3}$$

(d) Show by substitution that,

$$W(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{(x-x')^2}{2\sigma^2\tau}\right), \tag{4}$$

(where x' is an arbitrary constant) satisfies (3) with the condition,

$$\int_{-\infty}^{+\infty} W(x,\tau)\, dx = 1 \tag{5}$$

(e) Hence write down the solution of the Black-Scholes equation (1) which corresponds to (4).

(f) Briefly indicate how this result can be used to derive a specific solution, given boundary data at the maturity date.

Hint: $\int_{-\infty}^{+\infty} \exp(-\alpha x^2)\, dz = \sqrt{\frac{\pi}{\alpha}}$.

ANSWER

(a)

$$V(S,t) = e^{-t(T-t)}U(S,t)$$

$\frac{\partial V}{\partial t} = re^{-r(T-t)}U(S,t) + e^{-r(T_t)}\frac{\partial U}{\partial t}$
$\frac{\partial V}{\partial S} = e^{-r(T-t)}\frac{\partial U}{\partial S}$
$\frac{\partial^2 V}{\partial S^2} = e^{r(T_t)}\frac{\partial^2 U}{\partial S^2}$

Putting these into Black-Scholes

$$re^{-r(T-t)}U + e^{-r(T-t)}\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 e^{-r(T-t)}\frac{\partial^2 U}{\partial S^2} + rSe^{-r(T-t)}\frac{\partial U}{\partial S} - rS\frac{\partial U}{\partial S} = 0$$

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0 \tag{6}$$

(b)

$$\begin{aligned}
\tau = T - t \Rightarrow \frac{\partial U}{\partial t} &\rightarrow -\frac{\partial U}{\partial \tau} \\
\xi = \log S \Rightarrow s = e^{\xi} \Rightarrow \frac{\partial U}{\partial S} &= \frac{\partial \xi}{\partial S}\frac{\partial U}{\partial \xi} = \frac{1}{S}\frac{\partial U}{\partial \xi} = e^{-\xi}\frac{\partial U}{\partial \xi} \\
\frac{\partial^2 U}{\partial S^2} &= e^{-\xi}\frac{\partial}{\partial \xi}\left(e^{-\xi}\frac{\partial U}{\partial \xi}\right) \\
&= e^{-2\xi}\frac{\partial^2 U}{\partial \xi^2} - e^{-2\xi}\frac{\partial U}{\partial \xi}
\end{aligned}$$

Substituting this in (1)

$$-\frac{\partial U}{\partial \tau} + \frac{1}{2}\sigma^2 e^{2\xi}\left(e^{-2\xi}\frac{\partial^2 U}{\partial \xi^2} = e^{-2\xi}\frac{\partial U}{\partial \xi}\right) + re^{\xi}e^{-\xi}\frac{\partial U}{\partial \xi} = 0$$

Therefore

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi} \tag{7}$$

(c) Set $x = \xi\left(r - \frac{\sigma^2}{2}\right)\tau,\ \overline{\tau} = \tau$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial \overline{\tau}}{\partial \xi}\partial\partial\overline{\tau} = \frac{\partial}{\partial x} + 0 = \frac{\partial}{\partial x} \Rightarrow \frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial x^2}$$

$$\frac{\partial \tau}{= \partial \tau}\frac{\partial x}{\partial x}\frac{\partial}{} + \frac{\partial \overline{\tau}}{\partial \tau}\frac{\partial}{\partial \overline{\tau}} = \left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial \overline{\tau}}$$

Set $U = W(x, \overline{\tau})$

and substitute into (2)

$$\left(\left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial \overline{\tau}}\right)W = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial W}{\partial x}$$

Therefore

$$\frac{\partial W}{\partial \overline{\tau}} = \frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial x^2}$$

or

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2}$$

removing the irrelevant $(\tau \longrightarrow \overline{\tau})$ transformation.

(d)

$$W(x, \tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-x')^2}{2\sigma^2\tau}}$$

$$\begin{aligned}
\frac{\partial W}{\partial \tau} &= \frac{1}{\sigma\sqrt{2\pi\tau}} \times \left(\frac{(x-x')^2}{2\sigma^2\tau^2}\right) e^{-\frac{(x-x')^2}{2\sigma^2\tau} - \frac{1}{2\sigma}\frac{e^{-\frac{(x-x')^2}{2\sigma^2\tau}}}{\sqrt{2\pi\tau}^3}} \\
\frac{\partial W}{\partial x} &= \frac{1}{\sigma\sqrt{2\pi\tau}} \times \left(-\frac{2(x-x')}{2\sigma^2\tau}\right) e^{-\frac{(x-x')^2}{2\sigma^2\tau}} \\
\frac{\partial^2 W}{\partial x^2} &= \frac{1}{\sigma\sqrt{2\pi\tau}} \times \left[-\frac{1}{\sigma^2\tau} + \frac{(x-x')^2}{\sigma^4\tau^2}\right] e^{-\frac{(x-x')^2}{2\sigma^2\tau}}
\end{aligned}$$

Therefore

$$
\begin{aligned}
\frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2} &= \frac{\sigma}{2\sqrt{2\pi\tau}}\left[-\frac{1}{\sigma^2\tau}+\frac{(x-x')^2}{\sigma^4\tau^2}\right]e^{-\frac{(x-x')^2}{2\sigma^2\tau}} \\
&= -\frac{1}{2}\frac{e^{-\frac{(x-x')^2}{2\sigma^2\tau}}}{\sqrt{2\pi}\tau^{\frac{3}{2}}\sigma}+\frac{1}{\sigma\sqrt{2\pi\tau}}\frac{(x-x')^2}{2\sigma^2\tau^2}e^{-\frac{(x-x')}{2\sigma^2\tau}} \\
&= \frac{\partial W}{\partial t}
\end{aligned}
$$

(see above). Therefore

$$
W(x,\tau)=\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-x')^2}{2\sigma^2\tau}}
$$

is a solution of $\frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial kx^2}=\frac{\partial W}{\partial \tau}$

But

$$
\begin{aligned}
\int_{-\infty}^{+\infty} W(x,\tau)\,dx &= \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}\,dx \\
&= \frac{1}{\sigma\sqrt{2\pi\tau}}\sqrt{\frac{\pi}{\frac{1}{2\sigma^2\tau}}} \\
&= \frac{1}{\sigma\sqrt{2\pi\tau}}\sigma\sqrt{2\pi\tau} \\
&= 1
\end{aligned}
$$

Therefore this is the solution required.

(e) Back substitute the variables:

$$
x=\xi+\left(r-\frac{\sigma^2}{2}\right)\tau=\log S+\left(r-\frac{\sigma^2}{2}\right)(T-t)
$$

$$
\tau=T-t,\ U\leftrightarrow W
$$

therefore

$$
V(s,t)=e^{-r(T-t)}\times\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{\left[\log\left(\frac{s}{s'}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)\right]}{2\sigma^2(T-t)}}
$$

$\log s'=x'$ is required solution.

(f) Equation (4) is the fundamental solution of the diffusion equation (3), effectively a Green's-function (or derivative). To find specific solution can write it as

$$W(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{2\sigma^2\tau}} g(x')\, dx$$

where $g(x)$ is the boundary data for $\tau = 0$ (i.e. at maturity).

This satisfies (3) by differentiation under integral sign and also becomes a *delta*-function integration as $\tau \to 0$ therefore satisfying boundary data).

Feeding in substitutes we arrive at

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \exp\left\{ - \frac{\left(\log\left(\frac{s}{s'}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)} \right\}$$

$$\times \mathit{Payoff}\,(S')\frac{dS'}{S'}$$

where *payoff* (S)=payoff function at maturity $t = T$.