Question

Use Laplace transforms to solve the following systems of equations:

(a)

$$\begin{aligned} x' &= y + e^t \\ y' &= -2x + 3y + 2 \end{aligned}$$

with initial conditions $x(0) = 2$, $y(0) = 2$.

(b)

$$\begin{aligned} x' &= 2x + y + te^{2t} \\ y' &= -4x + 2y - e^{2t} \end{aligned}$$

with initial conditions $x(0) = 1$, $y(0) = 1$.

Answer

(a) Taking the Laplace transform of the equations gives:

$$sX - 2 = Y + \frac{1}{s-1} \qquad (1)$$

$$sY - 2 = -2X + 3Y + \frac{2}{s} \qquad (2)$$

From (1) we get $X = \dfrac{Y}{s} + \dfrac{1}{s(s-1)} + \dfrac{2}{s}$

Substituting in (2) gives $sY - 2 = -\dfrac{2Y}{s} - \dfrac{2}{s(s-1)} - \dfrac{4}{s} + 3Y + \dfrac{2}{s}$.

Rearranging and multiplying by s^2 gives

$(s^2 - 3s + 2)Y = 2s - \dfrac{2}{s-1} - 2$ which gives

$Y = \dfrac{2}{s-2} - \dfrac{2}{(s-1)^2(s-2)}$. On taking partial fractions this simplifies to

$Y = \dfrac{2}{s-1} + \dfrac{2}{(s-1)^2}$. Hence $y(t) = 2e^t + 2te^t$.

We now rewrite the second differential equation as $x = \dfrac{1}{2}(3y + 2 - y')$ and substitute for $y(t)$ to obtain

$x(t) = e^t + 2te^t + 1$.

(b) Taking the Laplace transform of the equations gives:

$$sX - 1 = 2X + Y + \frac{1}{(s-2)^2} \qquad (1)$$

$$sY - 1 = -4X + 2Y - \frac{1}{(s-2)} \qquad (2)$$

Rearranging gives

$$(s-2)X - Y = 1 + \frac{1}{(s-2)^2} \qquad (3)$$

$$4X + (s-2)Y = 1 - \frac{1}{(s-2)} \qquad (4)$$

$(s-2)(3) + (4) \quad \Rightarrow \quad [4 + (s-2)^2]X = 1 + s - 2$

$\Rightarrow \quad X = \frac{1}{2}\frac{2}{(s-2)^2+4} + \frac{s-2}{(s-2)^2+4}$

$\Rightarrow \quad x(t) = \frac{1}{2}e^{2t}\sin 2t + e^{2t}\cos 2t.$

But from the first differential equation $y = x' - 2x - te^{2t}$ and substituting for $x(t)$ gives

$y(t) = -2e^{2t}\sin 2t + e^{2t}\cos 2t - te^{2t}.$