Question

Solve each of the following second order equations by using Laplace transforms and check your answers.

(a) $y'' + 4y = 9t \qquad y(0) = 0, \qquad y'(0) = 7$

(b) $y''' + y = e^t \qquad y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 0$

(c) $y'' + 4y = 1 - H(t-1) \qquad y(0) = 0, \qquad y'(0) = 1$

(d) $y'' + 4y = \delta(t-2) \qquad y(0) = 0, \qquad y'(0) = 1$

Answer

(a) $y'' + 4y = 9t$, $y(0) = 0$, $y'(0) = 7$. Taking the Laplace transform gives

$$s^2Y - sy(0) - y'(0) + 4Y = \frac{9}{s^2} \quad \Rightarrow \quad (s^2+4)Y = \frac{9}{s^2} + 7$$

$$\begin{aligned} \Rightarrow \quad Y &= \frac{9}{s^2(s^2+4)} + \frac{7}{(s^2+4)} \\ &= \frac{9}{4}\frac{1}{s^2} - \frac{9}{4}\frac{1}{(s^2+4)} + \frac{7}{(s^2+4)} \\ &= \frac{9}{4}\frac{1}{s^2} + \frac{19}{8}\frac{2}{s^2+4} \end{aligned}$$

$$\Rightarrow \quad y(t) = \frac{9t}{4} + \frac{19}{8}\sin 2t.$$

(b) $y''' + y = e^t$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$. Taking the Laplace transform gives

$$s^3Y - s^2y(0) - sy'(0) - y''(0) + Y = \frac{1}{s-1}, \quad \Rightarrow \quad (s^3+1)Y = \frac{1}{s-1}.$$

Hence

$$Y = \frac{1}{(s^3+1)(s-1)} = \frac{1}{(s+1)(s^2-s+1)(s-1)} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{Cs+D}{s^2-s+1}.$$

Hence $A(s-1)(s^2-s+1) + B(s+1)(s^2-s+1) + (Cs+D)(s+1)(s-1) = 1$.

$\Rightarrow \quad (A+B+C)s^3 + (-2A+D)s^2 + (2A-C)s + (-A+B-D) = 1.$

Thus $A+B+C = 0$, $-2A+D = 0$, $2A-C = 0$ and $-A+B-D = 1$. Solving these equations gives

$A=-\frac{1}{6}, \quad B=\frac{1}{2}, \quad C=-\frac{1}{3}, \quad D=-\frac{1}{3}$. So that

$$
\begin{aligned}
Y(s) &= -\frac{1}{6}\frac{1}{(s+1)}+\frac{1}{2}\frac{1}{(s-1)}-\frac{1}{3}\frac{s+1}{s^2-s+1} \\
&= -\frac{1}{6}\frac{1}{(s+1)}+\frac{1}{2}\frac{1}{(s-1)}-\frac{1}{3}\frac{(s-1/2)+3/2}{(s-1/2)^2+(\sqrt{3}/2)^2} \\
&= -\frac{1}{6}\frac{1}{(s+1)}+\frac{1}{2}\frac{1}{(s-1)} \\
&\quad -\frac{1}{3}\frac{(s-1/2)}{(s-1/2)^2+(\sqrt{3}/2)^2}-\frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s-1/2)^2+(\sqrt{3}/2)^2}
\end{aligned}
$$

So that $y(t)=-\frac{1}{6}e^{-t}+\frac{1}{2}e^{t}-\frac{1}{3}e^{1/2t}\cos\frac{\sqrt{3}}{2}t-\frac{1}{\sqrt{3}}e^{1/2t}\sin\frac{\sqrt{3}}{2}t$.

(c) $y''+4y=1-H(t-1)$, $y(0)=0$, $y'(0)=1$. Taking the Laplace transform gives

$s^2Y-sy(0)-y'(0)+4Y=\frac{1}{s}-\frac{e^{-s}}{s}$ Hence

$(s^2+4)Y=\frac{1}{s}-\frac{e^{-s}}{s}+1$. Thus

$$
\begin{aligned}
Y &= \frac{1}{s(s^2+4)}-\frac{e^{-s}}{s(s^2+4)}+\frac{1}{s(s^2+4)} \\
&= \frac{1}{4}\frac{1}{s}-\frac{1}{4}\frac{s}{s^2+4}+\frac{1}{2}\frac{2}{s^2+4}-\frac{1}{4}\frac{e^{-s}}{s}+\frac{1}{4}\frac{se^{-s}}{s^2+4}
\end{aligned}
$$

$$\Rightarrow \quad y(t)=\frac{1}{4}-\frac{1}{4}\cos 2t+\frac{1}{2}\sin 2t-\frac{1}{4}H(t-1)+\frac{1}{4}H(t-1)\cos 2(t-1).$$

(d) $y''+4y=\delta(t-2)$, $y(0)=0$, $y'(0)=1$. Taking the Laplace transform gives

$s^2Y-sy(0)-y'(0)+4Y=e^{-2s}$, Hence $(s^2+4)Y=1+e^{-2s}$.

$$\Rightarrow \quad Y=\frac{1}{2}\frac{2}{s^2+4}+\frac{e^{-2s}}{2}\frac{2}{s^2+4}.$$

$$\Rightarrow \quad y(t)=\frac{1}{2}\sin 2t+\frac{1}{2}H(t-2)\sin 2(t-2).$$