Question

Find the Laplace transform of each of the following functions. In each case specify the values of s for which the transform exists.

(a) $6\sin 2t - 5\cos 2t$

(b) $(\sin t - \cos t)^2$

(c) $(t^2+1)^2$

(d) $t^3 e^{-2t}$

(e) $e^{-4t}\cosh 2t$

Answer

(a) $\mathcal{L}\{6\sin 2t - 5\cos 2t\} = \int_0^\infty e^{-st}\{6\sin 2t - 5\cos 2t\}\,dt$. Now

$$\begin{aligned} I = \int_0^\infty \sin 2t dt &= \left[-\frac{1}{s}e^{-st}\sin 2t\right]_0^\infty + \int_0^\infty \frac{2}{s}\cos 2t\,dt \\ &= 0 + \left[-\frac{2}{s^2}e^{-st}\cos 2t\right]_0^\infty + \int_0^\infty \frac{-4}{s^2}\sin 2t\,dt \\ &= \frac{2}{s^2} - \frac{4}{s^2}I \end{aligned}$$

Hence $\dfrac{s^2+4}{s^2}I = \dfrac{2}{s^2}$, which gives $I = \dfrac{2}{s^2+4}$.

Similarly $\int_0^\infty \cos 2t dt = \dfrac{s}{s^2+4}$, and hence $\mathcal{L}\{6\sin 2t - 5\cos 2t\} = \dfrac{12-5s}{s^2+4}$ for $s > 0$.

(b)

$$\begin{aligned} \mathcal{L}\{(\sin t - \cos t)^2\} &= \mathcal{L}\{\sin^2 t - 2\sin t\cos t + \cos^2 t\} \\ &= \mathcal{L}\{1 - \sin 2t\} \\ &= \frac{1}{s} - \frac{2}{s^2+4} \quad s > 0 \end{aligned}$$

(c) $\mathcal{L}\{(t^2+1)^2\} = \mathcal{L}\{t^4 + 2t^2 + 1\}$. Now $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$. So that

$$\mathcal{L}\{(t^2+1)^2\} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} = \frac{24+4s^2+s^4}{s^5}, \text{ for } s > 0.$$

(d) $\mathcal{L}\{t^3e^{-2t}\} = F(s+2)$ where $F(s) = \mathcal{L}\{t^3\} = \dfrac{6}{s^4}$.

Hence $\mathcal{L}\{t^3e^{-2t}\} = \dfrac{6}{(s+2)^4}$, for $s > -4$.

(e) $\mathcal{L}\{e^{-4t}\cosh 2t\} = F(s+4)$ where $F(s) = \mathcal{L}\{\cosh 2t\} = \dfrac{s}{s^2-4}$.

Hence $\mathcal{L}\{e^{-4t}\cosh 2t\} = \dfrac{s+4}{s^2+8s+12}$, for $s > -4$.