Question

Let S be the set of real numbers in $(0,1)$ whose decimal expansions do not contain the digit 4. Prove that $m^*(S) = 0$. Generalise this result as far as you can.

Answer

Consider the interval

$$I_{a_1\cdots a_n} = \left[\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots \frac{a_{n-1}}{10^{n-1}} + \frac{a_n}{10^n}, \frac{a_1}{10} + \cdots \frac{a_n + 1}{10^n}\right] = [\alpha, \beta]$$

$a_i \epsilon T$ where $T = 0, 1, 2, 3, 5, 6, 7, 8, 9$

Then $\displaystyle\bigcup_{\substack{i=1 \\ a_i \epsilon T}}^{n} I_{a_1\cdots a_n} \supseteq S$

For $\alpha \leq a_1 \cdots a_n a_{n+1} \cdots \leq \beta$

Thus $m^*(S) \leq \sum |I_{a_1\cdots a_n}| = 9^n \dfrac{1}{10^n} < \epsilon$ for $n > n_0$.