Question

We have defined m^* by $m^*(S) = \inf\left\{\sum_{i=1}^{\infty}|R_i| : \bigcup_{i=1}^{\infty} R_i \supseteq S\right\}$

Let n^*_δ be defined by

$n^*_\delta(S) = \inf\left\{\sum_{i=1}^{\infty}|R_i| : \bigcup_{i=1}^{\infty} R_i \supseteq S,\ |R_i| < \delta,\ i = 1, 2, \cdots\right\}$

Show that n^*_δ is a monotonic function of δ.

Define n^* by $n^*(S) = \lim_{\delta \to 0+} n^*_\delta(S)$

Show that $n^*(S) = m^*(S)$

Answer

If $\delta < \delta'$ then $n^*_\delta(S)$ is an infimum taken over a smaller set than $n^*_{\delta'}(S)$

Hence $n^*_\delta(S) \geq n^*_{\delta'}(S)$

Thus $n^*_\delta(S)$ tends to a limit as $\delta \to 0+$.

Now $n^*_\delta(S)$ is an infimum taken over a smaller set than $m^*(S)$. Therefore $m^*(S) \leq n^*_\delta(S) \leq n^*(S)$

Now for all ϵ there exists $\{R_i\}$ $\quad \sum_{i=1}^{\infty}|R_i| < m^*(S) + \epsilon$

Let $\delta > 0$ and choose each rectangle R_i into sub-rectangles $\{R_{ij}\}_{j=1}^{m_j}$ so that $|R_{ij}| < \delta$.

Then $|R_i| = \sum_{j=1}^{m_i}|R_{ij}| \quad \bigcup_j Rij = R_i$

So $\bigcup_{ij} R_{ij} \supseteq S \quad |R_{ij}| < \delta$.

Hence $n^*_\delta(S) \leq \sum|R_{ij}| = \sum|R_i| < m^*(S) + \epsilon$

Therefore $n^*_\delta(S) \leq m^*(S) + \epsilon \quad$ for all ϵ, for all δ

Therefore $n^*(S) \leq m^*(S) + \epsilon$

Therefore $n^*(S) \leq m^*(S)$

Therefore $n^*(S) = m^*(S)$