QUESTION

For each of the following matrices A find an orthogonal matrix P such that P^tAP is diagonal:

$$\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

ANSWER

First matrix:

Eigenvalue 6 normalised eigenvector $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$

Eigenvalue 11 normalised eigenvector $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Second matrix:

Eigenvalue 1 normalised eigenvector $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

Eigenvalue 3 normalised eigenvector $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Eigenvalue 7 normalised eigenvector $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Third matrix:

The slick way to do the question is as follows.

Clearly $\lambda = -1$ is an eigenvalue but rank $(A + I) = 1$, so null$(A + I) = 3$. Hence -1 is a triple eigenvalue and (as the matrix is symmetric) the eigenspace is three-dimensional. The other eigenvalue is 3 (use the fact that the sum of the eigenvalues equals the trace or notice that the sum of entries in each row equals 3).

The eigenvalue -1 has eigenspace $w + x + y + z = 0$ so three mutually orthogonal eigenvectors in this space are required. In general one can choose any three independant eigenvectors and then use Gram-Schmidt to get an orthogonal set, but in the present case it is not hard to spot three orthogonal vectors.

Eigenvalue -1; normalised eigenvectors $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

The eigenvalue 3 has normalised eigenvector $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$