Question

The series scavenger hunt: for each of the infinite series given below, do the following:

- Determine whether the series converges absolutely, converges conditionally, or diverges;
- if the series converges, determine its limit, where possible.

1. $\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$;
2. $\sum_{n=0}^{\infty}(1.01)^n$;
3. $\sum_{n=1}^{\infty}(\frac{e}{10})^n$;
4. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$;
5. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$;
6. $\sum_{n=1}^{\infty} \frac{1}{1+3^n}$;
7. $\sum_{n=2}^{\infty} \frac{10n^2}{n^3-1}$;
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{37n^3+3}}$;
9. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n}$;
10. $\sum_{n=2}^{\infty} \frac{2}{\ln(n)}$;
11. $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2+1}$;
12. $\sum_{n=1}^{\infty} \frac{n+2^n}{n+3^n}$;
13. $\sum_{n=2}^{\infty} \frac{1}{n^2\ln(n)}$;
14. $\sum_{n=1}^{\infty} \frac{n^3+1}{n^4+2}$;
15. $\sum_{n=1}^{\infty} \frac{1}{n+n^{3/2}}$;
16. $\sum_{n=1}^{\infty} \frac{10n^2}{n^4+1}$;
17. $\sum_{n=2}^{\infty} \frac{n^2-n}{n^4+2}$;
18. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$;

19. $\sum_{n=1}^{\infty} \frac{1}{3+5^n}$;

20. $\sum_{n=2}^{\infty} \frac{1}{n-\ln(n)}$;

21. $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{3^n}$;

22. $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$;

23. $\sum_{n=1}^{\infty} \frac{1}{n^{(1+\sqrt{n})}}$;

24. $\sum_{n=1}^{\infty} 1/(2^n(n+1))$;

25. $\sum_{n=1}^{\infty} n!/(n^2 e^n)$;

26. $\sum_{n=2}^{\infty} \sqrt{n}/(3^n \ln(n))$;

27. $\sum_{n=2}^{\infty}(2n)!/(n!)^3$;

28. $\sum_{n=1}^{\infty}(1-(-1)^n)/n^4$;

29. $\sum_{n=1}^{\infty}(2+\cos(n))/(n+\ln(n))$;

30. $\sum_{n=3}^{\infty} 1/(n\ln(n)\sqrt{\ln(\ln(n))})$;

31. $\sum_{n=1}^{\infty} n^n/(\pi^n n!)$;

32. $\sum_{n=1}^{\infty} 2^{n+1}/n^n$;

33. $\sum_{n=1}^{\infty}(-1)^{n-1}/\sqrt{n}$;

34. $\sum_{n=1}^{\infty} \cos(\pi n)/((n+1)\ln(n+1))$;

35. $\sum_{n=1}^{\infty}(-1)^n(n^2-1)/(n^2+1)$;

36. $\sum_{n=1}^{\infty}(-1)^n/(n\pi^n)$;

37. $\sum_{n=1}^{\infty}(-1)^n(20n^2-n-1)/(n^3+n^2+33)$;

38. $\sum_{n=1}^{\infty} n!/(-100)^n$;

39. $\sum_{n=3}^{\infty} 1/(n\ln(n)(\ln(\ln(n)))^2)$;

40. $\sum_{n=1}^{\infty}(1+(-1)^n)/\sqrt{n}$;

41. $\sum_{n=1}^{\infty} e^n \cos^2(n)/(1+\pi^n)$;

42. $\sum_{n=2}^{\infty} n^4/n!$;

43. $\sum_{n=1}^{\infty}(2n)!6^n/(3n)!$;

44. $\sum_{n=1}^{\infty} n^{100}2^n/\sqrt{n!}$;

45. $\sum_{n=3}^{\infty}(1+n!)/(1+n)!$;

46. $\sum_{n=1}^{\infty} 2^{2n}(n!)^2/(2n)!$;

47. $\sum_{n=1}^{\infty}(-1)^n/(n^2+\ln(n))$;

48. $\sum_{n=1}^{\infty}(-1)^{2n}/2^n$;

49. $\sum_{n=1}^{\infty}(-2)^n/n!$;

50. $\sum_{n=0}^{\infty} -n/(n^2+1)$;

51. $\sum_{n=1}^{\infty} 100\cos(n\pi)/(2n+3)$;

52. $\sum_{n=10}^{\infty}\sin((n+1/2)\pi)/\ln(\ln(n))$;

53. $\sum_{n=1}^{\infty}(2n)!/(2^{2n}(n!)^2)$;

54. $\sum_{n=1}^{\infty}(n/(n+1))^{n^2}$;

55. $\sum_{n=1}^{\infty} 1/(1+2+\cdots+n)$;

56. $\sum_{n=1}^{\infty}\ln(n)/(2n^3-1)$;

57. $\sum_{n=1}^{\infty}\sin(n)/n^2$;

58. $\sum_{n=1}^{\infty}(-1)^n(n-1)/n$;

59. $\sum_{n=1}^{\infty}(-1)^n2^{3n}/7^n$;

60. $\sum_{n=1}^{\infty}\cos(n)/n^4$;

61. $\sum_{n=1}^{\infty}(-1)^n3^n/(n(2^n+1))$;

62. $\sum_{n=1}^{\infty}(-1)^{n-1}n/(n^2+1)$;

63. $\sum_{n=2}^{\infty}(-1)^{n-1}/(n\ln^2(n))$;

64. $\sum_{n=1}^{\infty}(-1)^{n-1}2^n/n^2$;

65. $\sum_{n=1}^{\infty}(-1)^n\sin(\sqrt{n})/n^{3/2}$;

66. $\sum_{n=1}^{\infty} n^4e^{-n^2}$;

67. $\sum_{n=1}^{\infty}\sin(n\pi/2)/n$;

68. $\sum_{n=2}^{\infty} 1/(\ln(n))^8$;

69. $\sum_{n=13}^{\infty} 1/(n\ln(n)(\ln(\ln(n)))^p)$, where $p > 0$ is an arbitrary positive real number;

Answer

We make implicit use of the fact that convergence and absolute convergence are the same for series with positive terms.

1. **converges absolutely:** we could apply the ratio test, but we do not need to use such heavy machinary. Instead, we note that

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{2}\frac{1}{(1-2/3)} = \frac{3}{2},$$

since $\sum_{n=0}^{\infty} \frac{2^n}{3^n}$ is a convergent geometric series.

2. **diverges:** this is a geometric series, and since $1.01 > 1$, it is a divergent geometric series.

3. **converges absolutely:** this is a convergent geometric series, since $\frac{e}{10} < 1$, and it converges to

$$\sum_{n=1}^{\infty} \left(\frac{e}{10}\right)^n = \sum_{n=0}^{\infty} \left(\frac{e}{10}\right)^n - 1 = \frac{1}{1-e/10} - 1 = \frac{10}{10-e} - \frac{10-e}{10-e} = \frac{e}{10-e}.$$

4. **converges absolutely:** we use the second comparison test: since $n^2 + n + 1 > n^2$ for all $n \geq 1$, we have that $\frac{1}{n^2+n+1} < \frac{1}{n^2}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we have that $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges.

5. **diverges:** note that for $n \geq 1$, we have that $n \geq \sqrt{n}$, and so $n + \sqrt{n} \leq 2n$. Therefore, $\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$ for $n \geq 1$. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, its multiple $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, and hence by the first comparison test the series $\sum_{n=1}^{\infty} 1/(n+\sqrt{n})$ diverges.

6. **converges absolutely:** since $1 + 3^n > 3^n$ for all $n \geq 1$, we have that $\frac{1}{1+3^n} < \frac{1}{3^n}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges, the second convergence test yields that $\sum_{n=1}^{\infty} 1/(1+3^n)$ converges.

7. **diverges:** we'll use the limit comparison test: for large values of n, it seems that $\frac{10n^2}{n^3-1}$ behaves like a constant multiple of $\frac{1}{n}$, and in fact

$$\lim_{n\to\infty} \frac{10n^2/(n^3-1)}{1/n} = \lim_{n\to\infty} \frac{10n^3}{n^3-1} = 10 = L.$$

Since the limit exists and $0 < L = 10 < \infty$, and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test yields that $\sum_{n=2}^{\infty} 10n^2/(n^3-1)$ diverges.

8. **converges absolutely:** again we'll use the limit comparison test: for large values of n, it seems that $1/\sqrt{37n^3+3}$ behaves like $1/n^{3/2}$, and in fact

$$\lim_{n\to\infty} \frac{1/\sqrt{37n^3+3}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{37n^3+3}} = \lim_{n\to\infty} \frac{1}{\sqrt{37+3/n^3}} = \frac{1}{\sqrt{37}} = L.$$

Since the limit exists and $0 < L = \frac{1}{\sqrt{37}} < \infty$, and since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, the limit comparison test yields that $\sum_{n=1}^{\infty} 1/\sqrt{37n^3+3}$ converges.

9. **converges absolutely:** we start this one with a bit of algebra, namely

$$\frac{\sqrt{n}}{n^2+n} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}.$$

From note 1., we know that $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges, and so by the second comparison test, $\sum_{n=1}^{\infty} \sqrt{n}/(n^2+n)$ converges.

10. **diverges:** since $\ln(n) < n$ for all $n \geq 2$, we have that $\frac{1}{\ln(n)} > \frac{1}{n}$ for all $n \geq 2$, and so $\sum_{n=2}^{\infty} 2/\ln(n)$ diverges by the first comparison test, comparing it to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

11. **converges absolutely:** since $0 < \sin^2(n) \leq 1$ for all $n \geq 1$, we have that

$$0 < \frac{\sin^2(n)}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all $n \geq 1$. Since we are dealing with a series with positive terms and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by note 1., we have that $\sum_{n=1}^{\infty} \sin^2(n)/(n^2+1)$ converges by the second comparison test.

12. **converges absolutely:** for this series, we start with a bit of algebraic massage:

$$\frac{n+2^n}{n+3^n} < \frac{n+2^n}{3^n} < \frac{2^n+2^n}{3^n} = 2\left(\frac{2}{3}\right)^n.$$

So, the second comparison test, comparing with the convergent geometric series $2\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ yields that $\sum_{n=1}^{\infty}(n+2^n)/(n+3^n)$ converges.

13. **converges absolutely:** since $1/(n^2\ln(n)) < 1/n^2$ for $n \geq 3$, since $\ln(n) \geq 1$ for $n \geq 3$, we have by the second comparison test that $\sum_{n=2}^{\infty} 1/(n^2\ln(n))$ converges.

14. **diverges:** for large values of n, it seems that the n^{th} in the series is approximately $\frac{1}{n}$, and so we might guess that the series diverges by the limit comparison test. To check this guess, we need to evaluate

$$\lim_{n\to\infty} \frac{(n^3+1)/(n^4+2)}{1/n} = \lim_{n\to\infty} \frac{n^4+n}{n^4+2} = 1 = L.$$

Since the limit exists and since $0 < L = 1 < \infty$, and since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=1}^{\infty}(n^3+1)/(n^4+2)$ diverges by the limit comparison test.

15. **converges absolutely:** since $\frac{1}{n+n^{3/2}} < \frac{1}{n^{3/2}}$ for all $n \geq 1$ and since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by note 1., we have that $\sum_{n=1}^{\infty} 1/(n + n^{3/2})$ converges by the second comparison test.

16. **converges absolutely:** for large values of n, it seems that the n^{th} term in this series is approximately equal to $\frac{10}{n^2}$, and so we might guess that this series converges by use of the limit comparison test. To verify this guess, we calculate

$$\lim_{n\to\infty} \frac{10n^2/(n^4+1)}{10/n^2} = \lim_{n\to\infty} \frac{n^4}{n^4+1} = 1 = L.$$

Since the limit exists and since $0 < L = 1 < \infty$, and since $\sum_{n=1}^{\infty} \frac{10}{n^2}$ converges by note 1., we have that $\sum_{n=1}^{\infty} 10n^2/(n^4+1)$ converges by the limit comparison test.

17. **converges absolutely:** for large values of n, it seems again that the n^{th} term in this series is approximately equal to $\frac{1}{n^2}$, and so we might guess that this series converges by use of the limit comparison test. To verify this guess, we calculate

$$\lim_{n\to\infty} \frac{(n^2-n)/(n^4+2)}{1/n^2} = \lim_{n\to\infty} \frac{n^4-n^3}{n^4+2} = 1 = L.$$

Since the limit exists and since $0 < L = 1 < \infty$, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by note 1., we have that $\sum_{n=2}^{\infty}(n^2-n)/(n^4+2)$ converges by the limit comparison test.

18. **diverges:** for large values of n, it seems that the n^{th} term of this series is approximately equal to $\frac{1}{n}$, and so we might guess that this series then diverges by the limit comparison test. To verify this guess, we calculate

$$\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1+1/n^2}} = 1 = L.$$

Since the limit exists and since $0 < L = 1 < \infty$, and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by note 1., we have that $\sum_{n=2}^{\infty} 1/\sqrt{n^2+1}$ diverges by the limit comparison test.

19. **converges absolutely:** since

$$\frac{1}{3+5^n} < \frac{1}{5^n} = \left(\frac{1}{5}\right)^n,$$

and since $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$ converges, the second comparison test yields that $\sum_{n=1}^{\infty} 1/(3+5^n)$ converges.

20. **diverges:** first note that since $\ln(n) < n$ for all $n \geq 2$, this is a series of positive terms. Also, $n-\ln(n) < n$, and so $1/(n-\ln(n)) > 1/n$. Hence, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty} 1/(n-\ln(n))$ diverges, by the first comparison test.

21. **converges absolutely:** since $0 < \cos^2(n) \leq 1$ for all $N \geq 1$, we have that $\cos^2(n)/3^n < 1/3^n$. Since $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, we have by the second comparison test that $\sum_{n=1}^{\infty} \cos^2(n)/3^n$ converges.

22. **converges absolutely:** since $1/(2^n+3^n) < 1/2^n$ and since $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, the second comparison test yields that $\sum_{n=1}^{\infty} 1/(2^n+3^n)$ converges.

23. **converges absolutely:** since $1+\sqrt{n} \geq 2$ for $n \geq 1$, we have that $n^{1+\sqrt{n}} \geq n^2$ for $n \geq 1$, and so $1/n^{(1+\sqrt{n})} \leq 1/n^2$ for $n \geq 1$. Hence, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by note 1., we have by the second comparison test that $\sum_{n=1}^{\infty} 1/n^{(1+\sqrt{n})}$ converges.

24. **converges absolutely:** since $2^n(n+1) > 2^n$ for $n \geq 1$, we have that $1/(2^n(n+1)) < 1/2^n$ for $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, we have by the second comparison test that $\sum_{n=1}^{\infty} 1/(2^n(n+1))$ converges.

25. **diverges:** since factorials are involved, we first see whether the ratio test gives us any information, and so we evaluate

$$\lim_{n\to\infty} \frac{(n+1)!/((n+1)^2 e^{n+1})}{n!/(n^2 e^n)} = \lim_{n\to\infty} \frac{(n+1)!n^2 e^n}{n!(n+1)^2 e^{n+1}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \frac{n+1}{e} = \infty,$$

and since $\infty > 1$, the ratio test implies that $\sum_{n=1}^{\infty} n!/(n^2 e^n)$ diverges.

[Though it's not obvious how, we could also have applied the n^{th} term test for divergence, since for large values of n we have

$$\frac{n!}{n^2e^n} = \frac{(n-1)(n-2)!}{ne^n} = \frac{n-1}{n}\frac{n-2}{e}\cdots\frac{2}{e}\frac{1}{e^2} > \frac{n-1}{n}\frac{2}{e}\frac{1}{e^2} > \frac{1}{e^3}.$$

We simplified by noting that the middle terms $\frac{n-2}{e}, \ldots, \frac{3}{e}$ are all greater than 1 and that $\frac{n-1}{n} > \frac{1}{2}$ for n large. Hence, $\lim_{n\to\infty} \frac{n!}{n^2e^n} \neq 0$.]

26. **converges absolutely:** there is not an obvious comparison to make, and so we try the ratio test:

$$\lim_{n\to\infty} \frac{\sqrt{n+1}/(3^{n+1}\ln(n+1))}{\sqrt{n}/(3^n\ln(n))} = \lim_{n\to\infty} \frac{1}{3}\frac{\ln(n)}{\ln(n+1)}\sqrt{\frac{n+1}{n}} = \frac{1}{3},$$

since $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = 1$, for instance using l'Hopital's rule. Since $\frac{1}{3} < 1$, the ratio test yields that $\sum_{n=1}^{\infty} \sqrt{n}/(3^n\ln(n))$ converges.

27. **converges absolutely:** since there are factorials involved, we first try the ratio test:

$$\lim_{n\to\infty} \frac{(2(n+1))!/((n+1)!)^3}{(2n)!/(n!)^3} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^3} = 0 < 1,$$

and so the ratio test yields that $\sum_{n=2}^{\infty}(2n)!/(n!)^3$ converges.

28. **converges absolutely:** note that the numerator of each term is either 0 or 2, and so this is a series with non-negative terms. Also, $(1-(-1)^n)/n^4 < 2/n^4$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by note 1., and so by the second comparison test $\sum_{n=1}^{\infty}(1-(-1)^n)/n^4$ converges.

29. **diverges:** we start with a bit of algebraic simplification:

$$\frac{2+\cos(n)}{n+\ln(n)} \geq \frac{1}{n+\ln(n)} > \frac{1}{2n}.$$

(The first inequality holds since $2+\cos(n) \geq 2+(-1) = 1$ for all $n \geq 1$, and the second inequality holds since $\ln(n) < n$ for all $n \geq 1$, and so $n+\ln(n) < n+n = 2n$.) Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (as it is a constant multiple of the harmonic series), the first comparison test yields that $\sum_{n=1}^{\infty}(2+\cos(n))/(n+\ln(n))$ diverges.

30. **diverges:** for this one, we use the integral test. Set

$$f(x) = \frac{1}{x\ln(x)\sqrt{\ln(\ln(x))}},$$

so that $a_n = f(n)$ for all $n \geq 3$. (The restriction that $n \geq 3$ is to ensure that $\sqrt{\ln(\ln(n))}$ is well defined.) In order to apply the integral test, we need to know that $f(x)$ is decreasing, which involves calculating a derivative and checking its sign:

$$f'(x) = \frac{-\left(\ln(x)\sqrt{\ln(\ln(x))} + \sqrt{\ln(\ln(x))} + x\ln(x)\frac{1}{2\sqrt{\ln(\ln(x))}}\frac{1}{x\ln(x)}\right)}{(x\ln(x)\sqrt{\ln(\ln(x))})^2} < 0.$$

Hence, the integral test can be applied, and says that $\sum_{n=3}^{\infty} 1/(n\ln(n)\sqrt{\ln(\ln(n))})$ converges if and only if $\int_3^{\infty} f(x)\mathrm{d}x = \lim_{M\to\infty}\int_3^M f(x)\mathrm{d}x$ exists. So, we calculate:

$$\lim_{M\to\infty}\int_3^M f(x)\mathrm{d}x = \lim_{M\to\infty}\int_3^M \frac{1}{x\ln(x)\sqrt{\ln(\ln(x))}}\mathrm{d}x = \lim_{M\to\infty} 2\sqrt{\ln(\ln(x))}\,\Big|_3^M\,,$$

which diverges, and so $\sum_{n=3}^{\infty} 1/(n\ln(n)\sqrt{\ln(\ln(n))})$ diverges.

31. **converges absolutely:** try the ratio test, since there are factorials about:

$$\lim_{n\to\infty}\frac{(n+1)^{(n+1)}/(\pi^{(n+1)}(n+1)!)}{n^n/(\pi^n n!)} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\frac{1}{\pi} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\frac{1}{\pi} = \frac{e}{\pi} = L.$$

Since the limit exists and since $L < 1$, the ratio test yields that $\sum_{n=1}^{\infty} n^n/(\pi^n n!)$ converges.

32. **converges absolutely:** since both the numerator and the denominator are raised to (essentially) the same power, we try the root test, and so need to calculate:

$$\lim_{n\to\infty}\left(\frac{2^{n+1}}{n^n}\right)^{1/n} = \lim_{n\to\infty} 2^{1/n}\frac{2}{n} = L = 0$$

(since $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$). Since the limit exists and since $L < 1$, the root test yields that $\sum_{n=1}^{\infty} 2^{n+1}/n^n$ converges.

33. **converges conditionally:** we first test for absolute convergence, by considering the related series $\sum_{n=1}^{\infty} |(-1)^{n-1}/\sqrt{n}| = \sum_{n=1}^{\infty} 1/\sqrt{n}$, which diverges by note 1.

We now test for convergence. This is an alternating series, and so we use the alternating series test: write

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = (-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = (-1)\sum_{n=1}^{\infty}(-1)^n a_n,$$

where $a_n = \frac{1}{\sqrt{n}} > 0$ for all $n \geq 1$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and since $a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$ for all $n \geq 1$, the alternating series test applies and yields that this series converges.

Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

34. **converges conditionally:** we first check for absolute convergence, that is, convergence of the associated series $\sum_{n=1}^{\infty} |\cos(\pi n)/((n+1)\ln(n+1))| = \sum_{n=1}^{\infty} 1/((n+1)\ln(n+1))$. For this series, we apply the integral test, with $f(x) = 1/((x+1)\ln(x+1))$. Since

$$f'(x) = \frac{-\left(\ln(x+1) + (x+1)\frac{1}{x+1}\right)}{(x+1)^2(\ln(x+1))^2} = \frac{-(\ln(x+1)+1)}{(x+1)^2(\ln(x+1))^2} < 0$$

for $x \geq 1$, the integral test yields that the series converges if and only if $\int_1^{\infty} f(x)\mathrm{d}x = \lim_{M\to\infty} \int_1^M f(x)\mathrm{d}x$ exists, so we calculate:

$$\lim_{M\to\infty} \int_1^M \frac{1}{(x+1)\ln(x+1)}\mathrm{d}x = \lim_{M\to\infty} \ln(\ln(x+1)) \Big|_1^M = \lim_{M\to\infty} (\ln(\ln(M+1)) - \ln(\ln(2))),$$

which diverges (very very slowly). So, the series does not converge absolutely.

We now test for convergence. Since $\cos(\pi n) = (-1)^n$, this is an alternating series, and we start with the alternating series test. Since $(n+1)\ln(n+1) < (n+2)\ln(n+2)$ for all $n \geq 1$, we have that $1/((n+1)\ln(n+1)) > 1/((n+2)\ln(n+2))$ for $n \geq 1$. Since $\lim_{n\to\infty} 1/((n+1)\ln(n+1)) = 0$ (and since $1/((n+1)\ln(n+1)) > 0$ for $n \geq 1$), the alternating series test applies and yields that the series converges.

Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

35. **diverges:** since $\lim_{n\to\infty}(n^2-1)/(n^2+1) = 1$, we have that $\lim_{n\to\infty}(-1)^n(n^2-1)/(n^2+1)$ does not exist, and so $\sum_{n=1}^{\infty}(-1)^n(n^2-1)/(n^2+1)$ diverges by the n^{th} term test for divergence.

36. **converges absolutely:** we first test for absolute convergence, by considering the associated series $\sum_{n=1}^{\infty} |(-1)^n/(n\pi^n)| = \sum_{n=1}^{\infty} 1/(n\pi^n)$. Since $1/(n\pi^n) \leq 1/\pi^n$ for $n \geq 1$ and since $\sum_{n=0}^{\infty} \frac{1}{\pi^n}$ converges, the second comparison test yields that $\sum_{n=1}^{\infty} 1/(n\pi^n)$ converges, and hence that $\sum_{n=1}^{\infty}(-1)^n/(n\pi^n)$ converges absolutely.

37. **converges conditionally:** we first test for absolute convergence, that is, convergence of the associated series $\sum_{n=1}^{\infty} |(-1)^n(20n^2-n-1)/(n^3+n^2+33)| = \sum_{n=1}^{\infty}(20n^2-n-1)/(n^3+n^2+33)$. Since the n^{th} term looks like a constant multiple of $\frac{1}{n}$ for large n, let's try the limit comparison test:

$$\lim_{n\to\infty} \frac{(20n^2-n-1)/(n^3+n^2+33)}{1/n} = \lim_{n\to\infty} \frac{20n^3-n^2-n}{n^3+n^2+33} = 20 = L.$$

Since the limit exists and $0 < L < \infty$, the series being considered here diverges, since the harmonic series converges. So, the original series does not converge absolutely.

We now test for convergence. The series $\sum_{n=1}^{\infty}(-1)^n(20n^2-n-1)/(n^3+n^2+33) = \sum_{n=1}^{\infty}(-1)^n a_n$ is an alternating series, since $\frac{20n^2-n-1}{n^3+n^2+33} > 0$ for $n \geq 1$, and so let's check whether it satisfies the conditions of the alternating series test. Since $(20n^2-n-1)/(n^3+n^2+33)$ is a rational function and the denominator has higher degree than the numerator, we have that $\lim_{n\to\infty}(20n^2-n-1)/(n^3+n^2+33) = 0$. All that remains to check is whether the a_n are monotonically decreasing. For this, let $f(x) = (20x^2-x-1)/(x^3+x^2+33)$, so that $f(n) = a_n$, and check that it's decreasing, which involves calculating $f'(x)$:

$$f'(x) = \frac{-20x^4+2x^3+4x^2+1322x-33}{(x^3+x^2+33)^2} < 0$$

for all x greater than any of the roots of the numerator. So, the alternating series test applies, and yields that this series converges.

Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

38. **diverges:** note that, for $n \geq 101$, we have

$$\frac{n!}{100^n} = \frac{n(n-1)\cdots 1}{100^n} = \frac{n}{100}\frac{n-1}{100}\cdots\frac{101}{100}\frac{100}{100}\frac{99}{100}\cdots\frac{1}{100} > \frac{99}{100}\cdots\frac{1}{100},$$

and so $\lim_{n\to\infty} n!/(-100)^n$ does not exist. Hence, by the n^{th} term test for divergence, the series diverges.

39. **converges absolutely:** we apply the integral test, with the function $f(x) = \frac{1}{x\ln(x)(\ln(\ln(x)))^2}$. First, we check to see that $f(x)$ is decreasing, by calculating its derivative:

$$f'(x) = \frac{-(\ln(x)(\ln(\ln(x)))^2 + (\ln(\ln(x)))^2 + 1)}{(x\ln(x)(\ln(\ln(x)))^4} < 0$$

for $x \geq 2$ (and the denominator is non-zero for $x \geq 3$). So, now we need to calculate

$$\begin{aligned}\int_3^\infty f(x)\mathrm{d}x &= \lim_{M\to\infty}\int_3^M \frac{1}{x\ln(x)(\ln(\ln(x))^2}\mathrm{d}x \\ &= \lim_{M\to\infty}\frac{1}{\ln(\ln(x)}\Big|_3^M = \lim_{M\to\infty}\left(\frac{-1}{\ln(\ln(M))}+\frac{1}{\ln(\ln(3))}\right) = \frac{1}{\ln(\ln(3))}.\end{aligned}$$

Since the limit converges, $\sum_{n=3}^\infty 1/(n\ln(n)(\ln(\ln(n)))^2)$ converges absolutely.

40. **diverges:** we start with a bit of arithmetic, noting that the numerator satisfies: $(1+(-1)^n) = 0$ for n odd and $(1+(-1)^n) = 2$ for n even. Hence, the terms of the series are non-zero only for n even, so let's make the substitution $n = 2k$ for $k \geq 1$. Then, for n even, we have that

$$\frac{1+(-1)^n}{\sqrt{n}} = \frac{2}{\sqrt{2k}} = \frac{\sqrt{2}}{\sqrt{k}} > \frac{1}{\sqrt{k}}.$$

Hence, by the first comparison test and note 1., we have that $\sum_{n=1}^\infty(1+(-1)^n)/\sqrt{n}$ diverges.

41. **converges absolutely:** again, we begin with a bit of algebra, simplifying the n^{th} in the series by noting that

$$\frac{e^n\cos^2(n)}{1+\pi^n} \leq \frac{e^n}{1+\pi^n} \leq \frac{e^n}{\pi^n} = \left(\frac{e}{\pi}\right)^n,$$

where the first inequality follows from $\cos^2(n) \leq 1$ for all $n \geq 1$. Since $\sum_{n=0}^\infty(e/\pi)^n$ converges, the second comparison test yields that $\sum_{n=1}^\infty e^n\cos^2(n)/(1+\pi^n)$ converges.

42. **converges absolutely:** since there are factorials involved, let's first try the ratio test:

$$\lim_{n\to\infty}\frac{(n+1)^4/(n+1)!}{n^4/n!} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4\frac{n!}{(n+1)!} == \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4\frac{1}{n+1} = 0 = L.$$

Since the limit exists and since $L < 1$, the ratio test yields that the series converges.

43. **converges absolutely:** again, since there are factorials involved, we first try the ratio test:

$$\lim_{n\to\infty}\frac{(2(n+1))!6^{(n+1)}/(3(n+1))!}{(2n)!6^n/(3n)!} = \lim_{n\to\infty}\frac{6(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} = 0 = L.$$

Since the limit exists and since $L < 1$, the ratio test yields that the series converges.

44. **converges absolutely:** and yet again, since there are factorials involved, our first attempt should be with the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^{100} 2^{(n+1)}/\sqrt{(n+1)!}}{n^{100} 2^n/\sqrt{n!}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{100} \frac{2}{\sqrt{n+1}} = 0 = L.$$

Since the limit exists and since $L < 1$, the ratio test yields that this series converges.

45. **diverges:** since there are factorials involved, we first try the ratio test:

$$\lim_{n\to\infty} \frac{(1+(n+1)!)/(1+(n+1))!}{(1+n!)/(1+n)!} = \lim_{n\to\infty} \frac{1+(n+1)!}{(1+n!)(n+2)} = \lim_{n\to\infty} \frac{1/n! + n + 1}{(1/n! + 1)(n+2)} = 1,$$

and so the ratio test gives no information. (This discussion was put in to remind you that the ratio test doesn't always work with factorials.)

Hmm. Notice that when n is large, $1 + n!$ is very nearly equal to $n!$, and so $(1+n!)/(n+1)!$ is very nearly equal to $n!/(n+1)! = 1/(n+1)$. So, let's try the limit comparison test with $1/(n+1)$:

$$\lim_{n\to\infty} \frac{(1+n!)/(1+n)!}{1/(n+1)} = \lim_{n\to\infty} \frac{(n+1)(1+n!)}{(n+1)!} = \lim_{n\to\infty} \frac{1+n!}{n!} = 1 = L.$$

Since the limit exists and since $\sum_{n=0}^{\infty} 1/(n+1)$ diverges (as it's the harmonic series less the leading term), the series $\sum_{n=3}^{\infty}(1+n!)/(1+n)!$ diverges by the limit comparison test.

46. **diverges:** again, since there are factorials involved, we first try the ratio test:

$$\lim_{n\to\infty} \frac{2^{2(n+1)}((n+1)!)^2}{(2(n+1))!} \frac{(2n)!}{2^{2n}(n!)^2} = \lim_{n\to\infty} \frac{4(n+1)^2}{(2n+1)(2n+2)} = 1,$$

and so the ratio test yields no information.

So, let's explicitly try the n^{th} term test for divergence. We start with a bit of algebraic massage, namely:

$$2^{2n}(n!)^2 = (2^n \cdot n!)^2 = ((2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2)^2,$$

and so

$$\frac{2^{2n}(n!)^2}{(2n)!} = \frac{(2n)\cdot(2n)\cdot(2n-2)\cdot(2n-2)\cdots 2\cdot 2}{(2n)\cdot(2n-1)\cdot(2n-2)\cdot(2n-3)\cdots 2\cdot 1} = \frac{(2n)\cdot(2n-2)\cdots 2}{(2n-1)\cdot(2n-3)\cdots 1} > 1.$$

In particular, the limit $\lim_{n\to\infty} 2^{2n}(n!)^2/(2n)!$ cannot be zero, and so the n^{th} term test yields that $\sum_{n=1}^{\infty} 2^{2n}(n!)^2/(2n)!$ diverges.

47. **converges absolutely:** we first check for absolute convergence, namely the convergence of the series $\sum_{n=1}^{\infty} |(-1)^n/(n^2+\ln(n))| = \sum_{n=1}^{\infty} 1/(n^2+\ln(n))$. Since $n^2+\ln(n) > n^2$, we have that $1/(n^2+\ln(n)) < 1/n^2$, and so by the second comparison test, the series $\sum_{n=1}^{\infty} 1/(n^2+\ln(n))$ converges. That is, the original series $\sum_{n=1}^{\infty}(-1)^n/(n^2+\ln(n))$ converges absolutely.

48. **converges absolutely:** we begin with a bit of algebraic massage, noting that

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2^n} = \sum_{n=1}^{\infty} \frac{((-1)^2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n.$$

This is a convergent geometric series, converging to

$$\frac{1}{1-\frac{1}{2}} - 1 = 1.$$

(The subtraction of 1 arises from the fact that the starting index in this series is not 0, so that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \left(\frac{1}{2}\right)^0 = \sum_{n=0}^{\infty} \frac{1}{2^n} - 1 = 2 - 1 = 1.)$$

49. **converges absolutely:** we first check for absolute convergence, namely the convergence of the series $\sum_{n=1}^{\infty} |(-2)^n/n!| = \sum_{n=1}^{\infty} 2^n/n!$. Since there are factorials involved, we make use of the ratio test:

$$\lim_{n\to\infty} \frac{2^{(n+1)}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 = L.$$

Since this limit exists and satisfies $L < 1$, the ratio test yields that $\sum_{n=1}^{\infty} 2^n/n!$ converges, and hence that the original series $\sum_{n=1}^{\infty}(-2)^n/n!$ converges absolutely.

50. **diverges:** first, note that this is not an alternating series, but is a series with all non-positive terms. Hence, for this series, convergence and absolute convergence are equivalent, as they are for series with non-negative terms.

    Now, for n large, $n/(n^2+1)$ is approximately equal to $1/n$, and so let's try the limit comparison test with $\frac{1}{n}$. So, we calculate:

    $$\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 = L.$$

    Since the limit exists and since $0 < L = 1 < \infty$, and since $\sum_{n=1}^{\infty} -1/n$ diverges (as it is a constant multiple of the harmonic series), the limit comparison test yields that the series $\sum_{n=1}^{\infty} -n/(n^2+1)$ diverges.

51. **converges conditionally:** we start by noting that $\cos(n\pi) = (-1)^n$, and so this is an alternating series. So, we first check for absolute convergence, namely the convergence of the series $\sum_{n=1}^{\infty} |100\cos(n\pi)/(2n+3)| = \sum_{n=1}^{\infty} 100/(2n+3)$. Here, there are many tests that yield divergence, for instance we may use the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$:

    $$\lim_{n\to\infty} \frac{100/(2n+3)}{1/n} = \lim_{n\to\infty} \frac{100n}{2n+3} = 50 = L;$$

    since this limit exists and satisfies $0 < L = 50 < \infty$, and since the harmonic series diverges, the limit comparison test yields that $\sum_{n=1}^{\infty} 100/(2n+3)$ diverges.

    However, since $\frac{100}{2(n+1)+3} = \frac{100}{2n+5} < \frac{100}{2n+3}$ and since $\lim_{n\to\infty} \frac{100}{2n+3} = 0$, the alternating series test yields that $\sum_{n=1}^{\infty} 100\cos(n\pi)/(2n+3)$ converges.

    Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

52. **converges conditionally:** as before, we begin by simplifying the expression of each term. Here, note that $\sin((n+1/2)\pi) = (-1)^n$, and so this is an alternating series. As always, we first check for absolute convergence, namely the convergence of the series $\sum_{n=10}^{\infty} |\sin((n+1/2)\pi)/\ln(\ln(n))| = \sum_{n=10}^{\infty} 1/\ln(\ln(n))$. Since $n > \ln(\ln(n))$ for all $n \geq 10$, we have that $1/\ln(\ln(n)) > 1/n$ for all $n \geq 10$, and so the series $\sum_{n=10}^{\infty} 1/\ln(\ln(n))$ diverges by the first comparison test. That is, the original series does not converge absolutely.

We are now ready to determine convergence of the original series. As this is an alternating series, let's check whether the hypthoses of the alternating series test are satisfied. Since $1/\ln(\ln(n)) > 1/\ln(\ln(n+1))$ and since $\lim_{n\to\infty} 1/\ln(\ln(n)) = 0$ (since $\lim_{n\to\infty} \ln(\ln(n)) = \infty$), the alternating series test applies to this series, and yields that the series $\sum_{n=10}^{\infty} \sin((n+1/2)\pi)/\ln(\ln(n))$ converges.

Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

53. **diverges:** similar to the algebraic manipulation we performed on the series whose terms were the reciprocals of the terms in this series, we calculate:

$$\begin{aligned}
\frac{(2n)!}{2^{2n}(n!)^2} &= \frac{(2n)!}{(2^n n!)^2} \\
&= \frac{(2n)\cdot(2n-1)\cdot(2n-2)\cdot(2n-3)\cdots 2\cdot 1}{(2n)\cdot(2n)\cdot(2n-2)\cdot(2n-2)\cdots 2\cdot 2} \\
&= \frac{(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdots 4\cdot 2} \\
&= \frac{1}{2n}\frac{2n-1}{2n-2}\frac{2n-3}{2n-4}\cdots\frac{5}{4}\frac{3}{2} > \frac{1}{2n}.
\end{aligned}$$

Hence, since the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (as it is a constant multiple of the harmonic series), the first comparison test yields that $\sum_{n=1}^{\infty}(2n)!/(2^{2n}(n!)^2)$ diverges.

54. **converges absolutely:** since each term is a power, we first attempt to apply the root test, and so we calculate:

$$\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} = L.$$

Since the limit exists and since $L < 1$, the root test yields that $\sum_{n=1}^{\infty}(n/(n+1))^{n^2}$ converges.

55. **converges absolutely:** we begin with a bit of algebraic manipulation, namely noting that

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

for $n \geq 1$, and so

$$\frac{1}{1+2+\cdots+n} = \frac{2}{n(n+1)} < \frac{2}{n^2}$$

for $n \geq 1$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges, by note 1., the second comparison test yields that $\sum_{n=1}^{\infty} 1/(1+2+\cdots+n)$ converges.

56. **converges absolutely:** we begin with a bit of simplification, namely noting that

$$0 \leq \frac{\ln(n)}{2n^3-1} \leq \frac{n}{2n^3-1} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

for $n \geq 1$. (The first inequality follows since $\ln(n) \leq n$ for $n \geq 1$, while the second inequality follows since $2n^3 - 1 \geq n^3$ for $n \geq 1$.) Since $\sum_{n=1}^{\infty} 1/n^2$ converges by note 1., the second comparison test yields that $\sum_{n=1}^{\infty} \ln(n)/(2n^3-1)$ converges.

57. **converges absolutely:** note that this is not an alternating series, even though the terms are not all of the same sign (since $\sin(n)$ behaves a bit strangely). However, we still begin testing for convergence by testing for absolute convergence, namely the convergence of the series $\sum_{n=1}^{\infty} |\sin(n)/n^2|$. Since $|\sin(n)| \leq 1$ for all $n \geq 1$, and since $\sum_{n=1}^{\infty} 1/n^2$ converges by note 1., the second comparison test yields that $\sum_{n=1}^{\infty} \sin(n)/n^2$ converges absolutely.

58. **diverges:** since $\lim_{n\to\infty}(n-1)/n = 1$, we have that $\lim_{n\to\infty}(-1)^n(n-1)/n$ does not exist (since for large n, it is oscillating between numbers near 1 and numbers near -1). Since this limit does not exist, the n^{th} term test for divergence yields that $\sum_{n=1}^{\infty}(-1)^n(n-1)/n$ diverges.

59. **diverges:** we can rewrite this series as a geometric series, to whit:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{3n}}{7^n} = \sum_{n=1}^{\infty} \frac{(-8)^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{-8}{7}\right)^n.$$

Since $|-\frac{8}{7}| \geq 1$, this is a divergent geometric series.

60. **converges absolutely:** this is similar to a series we handled a few problems ago. Even though the terms are not of the same sign and are not of alternating signs, we still begin our check for convergence by checking for absolute convergence. Since $|\cos(n)/n^4| \leq 1/n^4$ (since $|\cos(n)| \leq 1$ for all $n \geq 1$) and since $\sum_{n=1}^{\infty} 1/n^4$ converges, the second comparison test yields that $\sum_{n=1}^{\infty} \cos(n)/n^4$ converges absolutely.

61. **diverges:** even though this is an alternating series, I personally feel the need to try the n^{th} term test first, since for n large, the dominant terms are the 3^n in the numerator and the 2^n in the demoninator, and

so I expect that the value of $3^n/(n(2^n+1))$ to be large for large values of n. Let's check this:

$$\frac{3^n}{n(2^n+1)} = \frac{3^n}{n\,2^n+n} > \frac{3^n}{n\,2^n+n\,2^n} = \frac{3^n}{2n\,2^n} = \left(\frac{3}{2}\right)^n \frac{1}{2n}.$$

Now, notice that $(3/2)^n > n$ for $n \geq 3$ (since $(3/2)^3 > 3$ and the derivative of $(3/2)^n - n$ is positive for $n \geq 3$), and so

$$\frac{3^n}{n(2^n+1)} > \left(\frac{3}{2}\right)^n \frac{1}{2n} > \frac{1}{2}$$

for $n \geq 3$. (So, not exactly large for large values of n, but big enough to do the trick.) Hence, the limit $\lim_{n\to\infty}(-1)^n 3^n/(n(2^n+1))$ does not exist (as it oscillates positive and negative and never settles down to 0), and so by the n^{th} term test for divergence, $\sum_{n=1}^{\infty}(-1)^n 3^n/(n(2^n+1))$ diverges.

62. **converges conditionally:** we first check for absolute convergence, namely the convergence of the series $\sum_{n=1}^{\infty}|(-1)^{n-1}n/(n^2+1)| = \sum_{n=1}^{\infty} n/(n^2+1)$. Since $n/(n^2+1) > n/(n^2+n^2) = 1/(2n)$ for all $n \geq 1$ and since $\sum_{n=1}^{\infty} 1/(2n)$ diverges (as it is a constant multiple of the harmonic series), the first comparison test yields that $\sum_{n=1}^{\infty} n/(n^2+1)$ diverges, and so the original series does not converge absolutely.

    As it is an alternating series, we can attempt to check convergence by seeing if we can apply the alternating series test. Since $\lim_{n\to\infty} n/(n^2+1) = 0$ and since $n/(n^2+1) > (n+1)/((n+1)^2+1)$ for all $n \geq 1$, the hypotheses of the alternating series test are met, and so $\sum_{n=1}^{\infty}(-1)^{n-1}n/(n^2+1)$ converges.

    Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

63. **converges absolutely:** we first check absolute convergence, namely the convergence of the series $\sum_{n=2}^{\infty}|(-1)^{n-1}/(n\ln^2(n))| = \sum_{n=2}^{\infty} 1/(n\ln^2(n))$. For this series, we use the integral test: set $f(x) = 1/(x\ln^2(x))$. We need to check that $f(x)$ is decreasing, which we do by calculating its derivative:

    $$f'(x) = \frac{-(\ln^2(x) + 2\ln(x))}{x^2\ln^4(x)} < 0$$

    for $x \geq 2$ (since $\ln(x) > 0$ for $x \geq 2$). We now calculate:

    $$\int_2^{\infty} f(x)\mathrm{d}x = \lim_{M\to\infty}\int_2^M \frac{1}{x\ln^2(x)}\mathrm{d}x$$

$$= \lim_{M\to\infty} \frac{-1}{\ln(x)} \Big|_2^M$$

$$= \lim_{M\to\infty} \left(\frac{-1}{\ln(M)} + \frac{1}{\ln(2)} \right) = \frac{1}{\ln(2)}.$$

Since this limit exists, the integral test yields that the series $\sum_{n=2}^{\infty} 1/(n\ln^2(n))$ converges, and hence that the original series $\sum_{n=2}^{\infty}(-1)^{n-1}/(n\ln^2(n))$ converges absolutely.

64. **diverges:** we apply the ratio test (note 2.), as this is a series with non-zero terms:

$$\lim_{n\to\infty} \left| \frac{(-1)^n 2^{(n+1)}/(n+1)^2}{(-1)^{n-1}2^n/n^2} \right| = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2 = L.$$

Since this limit exists and satisfies $L > 1$, the series $\sum_{n=1}^{\infty}(-1)^{n-1}2^n/n^2$ diverges.

65. **converges absolutely:** we first check for absolute convergence, namely the convergence of the series $\sum_{n=1}^{\infty} |(-1)^n \sin(\sqrt{n})/n^{3/2}| = \sum_{n=1}^{\infty} |\sin(\sqrt{n})|/n^{3/2}$. Since $|\sin(\sqrt{n})|/n^{3/2} \leq 1/n^{3/2}$ for $n \geq 1$ (since $|\sin(\sqrt{n})| \leq 1$ for $n \geq 1$), and since $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges by note 1., the second comparison test yields that $\sum_{n=1}^{\infty} |\sin(\sqrt{n})|/n^{3/2}$ converges, and hence that the original series $\sum_{n=1}^{\infty}(-1)^n \sin(\sqrt{n})/n^{3/2}$ converges absolutely.

66. **converges absolutely:** even though there are no factorials, let us apply the ratio test. So, we calculate:

$$\lim_{n\to\infty} \frac{(n+1)^4 e^{-(n+1)^2}}{n^4 e^{-n^2}} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^4 e^{-2n-1} = 0 = L.$$

Since this limit exists and since $L < 1$, the ratio test yields that the series $\sum_{n=1}^{\infty} n^4 e^{-n^2}$ converges.

67. **converges conditionally:** before testing for absolute convergence, we perform a bit of algebraic simplification, by noting that

$$\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{2k\pi}{2}\right) = \sin(k\pi) = 0$$

for n even and

$$\sin\left(\frac{\pi n}{2}\right) = \sin\left(\frac{\pi(2k+1)}{2}\right) = \sin\left(k\pi + \frac{\pi}{2}\right) = (-1)^k$$

for $n = 2k+1$ odd. Hence, setting $n = 2k+1$ for $k \geq 0$, we may rewrite the series as

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} = \sum_{k=0}^{\infty} \frac{\sin(\pi(2k+1)/2)}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

We first test for absolute convergence, namely the convergence of the series $\sum_{k=0}^{\infty} |(-1)^k/(2k+1)| = \sum_{k=0}^{\infty} 1/(2k+1)$. However, since $1/(2k+1) > 1/(2k+2) = 1/2(k+1)$ and since $\sum_{k=0}^{\infty} 1/(k+1)$ is the harmonic series, the series $\sum_{k=0}^{\infty} 1/(2k+1)$ diverges by the first comparison test, and hence the original series does not converge absolutely.

To test convergence, we use the alternating series test. Since $1/(2k+1) > 1/(2(k+1)+1)$ for all $k \geq 0$ and since $\lim_{k\to\infty} 1/(2k+1) = 0$, the alternating series test yields that $\sum_{k=0}^{\infty}(-1)^k/(2k+1)$ converges.

Hence, this series converges but does not converge absolutely. That is, the series converges conditionally.

68. **diverges:** for this series, we first note that $\ln(x) < x^{1/8}$ for x large ($x > e^{32}$ works), as follows: consider the function $f(x) = x^{1/8} - \ln(x)$, and note that

$$f(e^{8k}) = (e^{8k})^{1/8} - \ln(e^{8k}) = e^k - 8k,$$

and so $f(e^{32}) = e^4 - 32 = 22.5982... > 0$.

Moreover, for $x \geq e^{32}$, we have that $f(x)$ is increasing: differentiating, we see that

$$f'(x) = \frac{1}{8}x^{-7/8} - \frac{1}{x} = \frac{1}{x}\left(\frac{1}{8}x - 1\right),$$

and so $f'(x) > 0$ for $x > 8$.

So, for $n > e^{32}$, we have that

$$\frac{1}{\ln(n)^8} > \frac{1}{(n^{1/8})^8} = \frac{1}{n},$$

and hence by the first comparison test, $\sum_{n=2}^{\infty} 1/(\ln(n))^8$ diverges. (Note that we are making heavy use of Note 3., that ignoring finitely many terms of a series does not affect its convergence or divergence.)

69. **converges absolutely:** (note that the lower limit 13 for the series yields that $\ln(n)$ and $\ln(\ln(n))$ are positive for all terms in the series.) We apply the integral test, using the function

$$f(x) = \frac{1}{x\ln(x)(\ln(\ln(x)))^p}.$$

We first check that $f(x)$ is decreasing:

$$f'(x) = \frac{-(\ln(x)\ln(\ln(x))^p + \ln(\ln(x))^p + p)}{(x\ln(x)\ln(\ln(x))^p)^2} < 0$$

for $x > 13$, since both $\ln(x) > 0$ and $\ln(\ln(x)) > 0$ for $x > 13$ and since $p > 0$ by assumption.

In order to apply the integral test, we now need to calculate:

$$\int_{13}^{\infty} f(x)\mathrm{d}x = \lim_{M\to\infty}\int_{13}^{M} \frac{1}{x\ln(x)(\ln(\ln(x)))^p}.$$

There are two cases: if $p = 1$, we get

$$\begin{aligned}\lim_{M\to\infty}\int_{13}^{M}\frac{1}{x\ln(x)(\ln(\ln(x))} &= \lim_{M\to\infty} \ln(\ln(\ln(x)))\Big|_{13}^{M} \\ &= \lim_{M\to\infty} (\ln(\ln(\ln(M))) - \ln(\ln(\ln(13)))) = \infty,\end{aligned}$$

and so for $p = 1$ the series diverges.

For $p \neq 1$, we get:

$$\begin{aligned}\lim_{M\to\infty}\int_{13}^{M}\frac{1}{x\ln(x)(\ln(\ln(x))^p} &= \lim_{M\to\infty}\frac{1}{-p+1}\frac{1}{\ln(\ln(x))^{p-1}}\Big|_{13}^{M} \\ &= \frac{1}{-p+1}\lim_{M\to\infty}\left(\ln(\ln(M))^{-p+1} - \ln(\ln(13))^{-p+1}\right),\end{aligned}$$

which converges for $p > 1$ (since $-p+1 < 0$)and diverges for $p < 1$ (since $-p+1 > 0$). Hence, the series $\sum_{n=13}^{\infty} 1/(n\ln(n)(\ln(\ln(n)))^p)$ converges if and only if $p > 1$. (Note that this is really just Note 1. in a bit of disguise.)

Note 1.
The series $\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges if and only if $s > 1$.

For $s = 1$, this series is called the **harmonic series**, and we can prove directly that it diverges. Note that $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$, that $\frac{1}{5} + \cdots + \frac{1}{8} > 4\frac{1}{8} = \frac{1}{2}$, and in general that

$$\frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \cdots + \frac{1}{2^k} > 2^{k-1}\frac{1}{2^k} = \frac{1}{2}.$$

Hence, the $(2^k)^{th}$ partial sum S_{2^k} satisfies $S_{2^k} > 1 + k\frac{1}{2}$. Since the terms in the harmonic series are all positive, the sequence of partial sums is monotonically increasing, and by the calculation done the sequence of partial sums is unbounded, and so the sequence of partial sums diverges. Hence, the harmonic series diverges.

Note 2.
Ratio and root tests for general series: Let $\sum_{n=0}^{\infty} a_n$ be a series with non-zero terms, so that $a_n \neq 0$ for all n.

- **Ratio test:** Suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists. If $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges. If $L = 1$, this test gives no information.
- **Root test:** Suppose that $\lim_{n\to\infty}(|a_n|)^{1/n} = L$ exists. If $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges. If $L = 1$, this test gives no information.

Note 3.
Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two infinite series, and suppose there exists P so that $a_n = b_n$ for all $n > P$. (That is, assume the terms of the two series are equal after some point.) Then, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} b_n$ converges. That is, the convergence or divergence of a series is not affected by mucking about with finitely many terms of the series.