Question
In each of the following, $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms.

1. Prove that, if $\{c_n\}$ is a sequence of positive terms satisfying $\lim_{n\to\infty} c_n = 0$, then $\sum_{n=1}^{\infty} a_n c_n$ converges;

2. Prove that, if $\{c_n\}$ is a sequence of positive terms satisfying $\lim_{n\to\infty} c_n = c \neq 0$, then $\sum_{n=1}^{\infty} a_n c_n$ converges.

Answer
Let $S_k = \sum_{n=1}^{k} a_n$ be the k^{th} partial sum of $\sum_{n=1}^{\infty} a_n$.

1. Since $\lim_{n\to\infty} c_n = 0$ and $c_n > 0$ for all n, there exists $M > 0$ so that $0 < c_n < 1$ for $n > M$. Let $M = \max(1, c_1, c_2, \ldots, c_M)$, and note that $c_n \leq M$ for all $n \geq 1$. In particular, the k^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n c_n$ satisfies

$$\sum_{n=1}^{k} a_n c_n \leq \sum_{n=1}^{k} a_n M = M\, S_k.$$

   Hence, the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n c_n$ forms a monotonic (since all the a_n and c_n are positive), bounded (above by $M \sum_{n=1}^{\infty} a_n$, below by 0) sequence, and so converges. That is, $\sum_{n=1}^{\infty} a_n c_n$ converges.

2. The proof in the case that $\lim_{n\to\infty} c_n = c \neq 0$ is very similar to the proof in the case that $\lim_{n\to\infty} c_n = 0$. Since $\lim_{n\to\infty} c_n = c \neq 0$, there exists $M > 0$ so that $c_n < c+1$ for $n > M$. Let $M = \max(c+1, c_1, c_2, \ldots, c_M)$, and note that $c_n \leq M$ for all $n \geq 1$. In particular, the k^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n c_n$ satisfies

$$\sum_{n=1}^{k} a_n c_n \leq \sum_{n=1}^{k} a_n M = M\, S_k.$$

   Hence, the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n c_n$ forms a monotonic (since all the a_n and c_n are positive), bounded (above by $M \sum_{n=1}^{\infty} a_n$, below by 0) sequence, and so converges. That is, $\sum_{n=1}^{\infty} a_n c_n$ converges.