Exam Question
Topic: CriticalPoints
Find and classify the critical points of the function

$$f(x,y) = 3x^4 + 12xy + 4y^3.$$

Calculate the value of the function at each of the critical points.

Solution

$$f(x,y) = 3x^4 + 12xy + 4y^3;\ f_x = 12x^3 + 12y;\ f_y = 12x + 12y^2$$

So the partial derivatives are zero when $x^3 + y = 0;\ x + y^3 = 0$.
Substituting for x gives $-y^6 + y = 0$ i.e. $y(1 - y^5) = 0$.
The only real solutions are $y = 0,\ y = 1$.
When $y = 0,\ x = 0$ and when $y = 1,\ x = -1$.
Calculating the second partial derivatives gives

$$f_{xx} = 36x^2;\ f_{yy} = 24y;\ f_{xy} = 12.\ \text{So}\ \Delta = f_{xy}^2 - f_{xx}f_{yy} = 144 - 864x^2y$$

So at $(0,0), \Delta = 144 > 0$ so $(0,0)$ is a saddle point.
At $(-1,1), \Delta = 144 - 864 < 0$ so $(-1,1)$ is a local minimum ($f_{xx} > 0$).
The values of f at the critical points are $f(0,0) = 0;\ f(-1,1) = -5$.