In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

The smooth curve $y(x)$ is defined for $-\log 2 \le x \le \log 2$ and is such that $y(\pm \log 2) = \frac{5}{4}$. If the curve is rotated about the x-axis, show that the area of the surface of revolution thus generated is given by

$$A = 2\pi \int_{-\log 2}^{\log 2} y\sqrt{1 + y'^2}\, dx$$

Hence show that the extremal y satisfies

$$\frac{y}{\sqrt{1 + y'^2}} = const.$$

and thus the area is stationary if $y = \cosh x$.

Now consider a surface of rotation in the shape of a cylindrical spool formed from two parallel discs of radius $\frac{5}{4}$ placed at $x = \pm \log 2$, joined along the x axis by an infinitely thin rod. By simple geometry, show that the surface area of this shape if given by $\frac{25\pi}{8}$ and is thus less than the apparent minimum value obtained with $y = \cosh x$. How do you explain this apparent contradiction? (Hint: remember the assumptions on differentiability that have been made in the first part of the question.)

Answer

PICTURE

Standard calculus gives
Surface area $\underbrace{dA}_{\text{elemental surface area}} = \underbrace{2\pi y}_{\text{circumference}} \underbrace{ds}_{\text{element width}}$

Thus

$$
\begin{aligned}
A &= \int 2\pi y \, ds \\
&= 2\pi \int y\sqrt{1+y'^2}\, dx
\end{aligned}
$$

standard results, see lecture notes
$A = 2\pi \int_{x=-\log 2}^{x=+\log 2} y\sqrt{1+y'^2}\, dx$ as required

$F = F(y, y')$ only, so E-L equation has first integral $y' \dfrac{\partial F}{\partial y'} - F = const$

$$
\begin{aligned}
& \frac{\partial F}{\partial y'} = \frac{2\pi y y'}{\sqrt{1+y'^2}} \\
\Rightarrow \quad & \frac{2\pi y y'^2}{\sqrt{1+y'^2}} - 2\pi y\sqrt{1+y'^2} = const \\
\Rightarrow \quad & \frac{2\pi y}{\sqrt{1+y'^2}} = const \\
\Rightarrow \quad & \frac{y}{\sqrt{1+y'^2}} = const = \alpha \text{say}
\end{aligned}
$$

Thus we have $y'^2 = \dfrac{y^2}{\alpha^2} - 1$ which is solved via standard integrals to give

$$y = \alpha \cosh\left(\frac{x}{\alpha} + c\right) \text{ for constant c, alpha}$$

¿From symmetry of boundary conditions, we need $c = 0$. The other condition is satisfied with $\alpha = 1$. Thus $\underline{y = \cosh x}$ is the extremal solution.
Surface of rotation
PICTURE

Surface area (inside $-\log 2 < x < \log 2$) is $2 \times \left[\pi \times \left(\frac{5}{4}\right)^2\right] = \frac{25\pi}{8} =$ 9.817...(A)
Now on extremal $y = \cosh x$ we have

$$\begin{aligned} A &= 2\pi \int_{-\log 2}^{+\log 2} \cosh^2 x \, dx \\ &= 2\pi \left[\log 2 + \frac{1}{2}\sinh(2\log 2)\right] \\ &= 10.25 \quad (B) \end{aligned}$$

Clearly $(B) > (A)$, but we have assumed that $y = \cosh x$ is a minimum. Assuming is still is (can be confirmed by considering second variation) there is an apparent contradiciton. The resolution is that the disc case is not formed by the rotation of a smooth function as we have assumed in the case of (B). Thus $y = \cosh x$ is the minimal result if we restrict the solution to smooth functions as in the question, the disc result is not a valid solution.Hence no paradox!