In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

The speed of light in a medium depends on a quantity ϵ and is given by $v = \frac{C}{\sqrt{\epsilon}}$ where c is constant. A ray of light travels from $A = (a, \alpha)$ to $B = (b, \beta)$ is a medium where the speed varies with height only, i.e. $\epsilon = \epsilon(y)$. Justify briefly that the time taken for the ray to travel an infinitesimal distance ds at height y is given by $dt = \frac{ds}{v(y)}$. Hence show that the total time taken to travel between A and B is

$$t = \int_A^B \frac{ds}{v(y)} = \int_a^b dx\sqrt{1+y'^2}\left(\frac{\sqrt{\epsilon(y)}}{c}\right) = \int_a^b dx F(y, y')$$

If the ray travels in the least possible time between these points, using the appropriate special case of the Euler equation show that the ray's path $y = y(x)$ satisfies $K^2(1+y'^2) = \epsilon$ for some constant K. Deduce that if $\epsilon = \epsilon(y)$, the ray travels in a parabola with the axis vertical.

Answer

In general for distance travelled ds in dt we have $\frac{ds}{dt} = V$ where V is the speed

$\Rightarrow dt = \frac{ds}{V(y)}$ is V just depends on y explicitly

Thus we have total time of flight

$$\begin{aligned} \int_0^t dt &= \int_A^B \frac{ds}{V(y)} - \\ \Rightarrow \qquad t &= \int_A^B \frac{ds}{V(y)} = \int_{x=a}^{x=b} dx \frac{\sqrt{1+y'^2}}{V(y)} \\ \Rightarrow \qquad t &= \int_a^b dx \underbrace{\sqrt{1+y'^2}\left(\frac{\sqrt{\epsilon(y)}}{c}\right)} \end{aligned}$$

Clearly this is only an explicit function of $F = F(y, y')$

if the rays follow the path of least time, use the E-L equation for $F = F(y, y')$:

$y'\dfrac{\partial F}{\partial y'} - F = const$

$\dfrac{\partial F}{\partial y'} = \dfrac{y'}{\sqrt{1+y'^2}}\dfrac{\sqrt{\epsilon(y)}}{c}$

Therefore $\dfrac{y'^2\sqrt{\epsilon(y)}}{\sqrt{1+y'^2}}c - \sqrt{1+y'^2}\dfrac{\sqrt{\epsilon(y)}}{c} = const$

$\Rightarrow \dfrac{\sqrt{\epsilon(y)}}{c\sqrt{1+y'^2}} = const$

$\Rightarrow \underline{\epsilon(y) = (1+y'^2)K^2}$ for some constant K.

Thus $\epsilon(y) = (1+y'^2)K^2 = y$ from question

$\Rightarrow Ky' = \pm\sqrt{y-K^2}$

$\Rightarrow \underline{y = K^2 + \dfrac{(x+c)^2}{4K^2}}$ (by standard integrals)

which as a parabola as required.