In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

The Principle of Least Action (PLA) states that the motion of a particle of mass m between points P and Q is a potential field $V(x, y)$ is such that, for $h = const$, and ds being the distance element along the particle's path, the action

$$A = \int_P^Q ds\sqrt{2m(h - V)}$$

is minimised. Show that if $V = mgy(x)$ (a vertical gravitational potential field) and the particle is under moves horizontally between $x = a$ and $x = b$ then the PLA requires minimisation of the following functional integral:

$$A = \int_a^b dx\sqrt{2m(h - mgy)(1 + y'^2)}.$$

Determine the particle path, given that $m = 1,\ g = 10,\ h = 1,\ a = 0,\ b = 1$ (in appropriate units). Sketch the path.

Answer

$$\begin{aligned} A &= \int_P^Q ds\sqrt{2m(h - V)} \\ &= \int_{x=a}^{x=b} dx\sqrt{1 + y'^2}\sqrt{\underbrace{2m(h - V)}} \\ &= \int_a^b dx\sqrt{2m(h - mgy)(1 + y'^2)} \end{aligned}$$

Set $m = 1,\ g = 10,\ h = 1,\ a = 0,\ b = 1$ and we have

$$A = \int_0^1 dx\sqrt{2(1 - 10y)(1 + y'^2)}$$

So $F = \sqrt{2(1 - 10y)(1 + y'^2)} = F(y, y')$ <u>only</u>

So E-L becomes $y'\dfrac{\partial F}{\partial y'} - F = const$

$\Rightarrow y'\dfrac{\sqrt{2(1-10y)}}{\sqrt{1+y'^2}}y' - \sqrt{2(1-10y)(1+y'^2)} = const$

$\Rightarrow \dfrac{\sqrt{2(1-10y)}}{\sqrt{1+y'^2}} = const = \sqrt{2}c$ say

Therefore $\dfrac{1-10y}{1+y'^2} = c^2 \Rightarrow y'^2 = \left(\dfrac{1}{c^2} - 1\right) - \dfrac{10y}{c^2}$

which integrates to give

$$\int \frac{dy}{\sqrt{\left(\frac{1}{c^2} - 1\right) - \dfrac{10y}{c^2}}} = \int dx$$

$\Rightarrow -2\dfrac{c^2}{10}\sqrt{\left(\dfrac{1}{c^2} - 1\right) - \dfrac{10y}{c^2}} = x + d \; d = const$

$\Rightarrow y = \dfrac{(1-c^2)}{10} - \dfrac{10}{4c^2}(x+d)^2$

A parabola. Would need boundary conditions to evaluate c and d.