In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

A particle moves along the y-axis and is at $y = 0$ when $x = 0$. It reaches $y = a$ when $x = T$. The motion is governed by Hamilton's principle which states that the particle moves so as to make

$$\frac{1}{2}\int_0^T dx(y'^2 + \omega^2 y^2)$$

stationary, where ω is a given constant. Show that the motion is $y = a\frac{\sinh \omega x}{\sinh \omega T}$. By also computing the second variation, show that it is independent of the path and also positive for all possible variations. Hence deduce that this is a minimum extremal.

Answer

Let $I = \frac{1}{2}\int_0^T dx(y'^2 + \omega^2 y^2)$

$F = y'^2 + \omega^2 y^2$

$\frac{\partial F}{\partial y} = 2\omega^2 y;\ \frac{\partial F}{\partial y'} = 2y'$

Therefore E-L is $2\omega^2 y - \frac{d}{dx}(2y') = 0$

$\Rightarrow \omega^2 - y - y'' = 0 \quad \omega = const$

Which can be solved easily to give

$y = A\cosh \omega x + B\sinh \omega x$

consts A and B from boundary conditions: $y(0) = 0,\ y(T) = a$

$\Rightarrow A = 0,\ B = \frac{a}{\sinh \omega t}$

$\Rightarrow \underline{y = \frac{a\sinh \omega x}{\sinh \omega T}}$

By computing second variation

$$
\begin{aligned}
V_2 &= \int_0^T \left[\eta'^2 \frac{\partial^2 F}{\partial y'^2} + 2\eta\eta' \frac{\partial^2 F}{\partial y \partial y'} + \eta^2 \frac{\partial^2 F}{\partial y^2}\right] \\
&= \int_0^T [2\eta'^2 + 0 + 2\omega^2\eta^2] \\
&= 2\int_0^T \, dx[\eta'^2 + \omega^2\eta^2]
\end{aligned}
$$

This is independent of the path $y(x)$ and is also positive for <u>all</u> possible $\eta(x) \neq 0$. Thus we have a weak minimum (from lecture notes)