In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

Find the extremals for

$$I(y,z) = \int_0^{\frac{\pi}{2}} dx(y'^2 + z'^2 + 2yz)$$

subject to $y(0) = z(0) = 0,\ y\left(\frac{\pi}{2}\right) = z\left(\frac{\pi}{2}\right) = 1.$

Answer

This is a (generalisation 2) type problem with

$$F = (y'^2 + z'^2 + 2yz) = F(y, y', z, z')$$

Thus $\frac{\partial F}{\partial y'} = 2y',\ \frac{\partial F}{\partial y} = 2z,\ \frac{\partial F}{\partial z'} = 2z',\ \frac{\partial F}{\partial z} = 2y$ etc.

and we have simultaneous E-l equations:

$$\left\{\begin{array}{rcl} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) & = & 0 \\ \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) & = & 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 2z - \frac{d}{dx}(2y') & = & 0 \\ 2y - \frac{d}{dx}(2z') & = & 0 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{rcl} y'' - z & = & 0 \\ z'' - y & = & 0 \end{array}\right\} \underset{\frac{d^2}{dx^2}}{\Rightarrow} \left\{\begin{array}{rcl} y^{(iv)} - z'' & = & 0 \\ z^{(iv)} - y'' & = & 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y^{(iv)} - y & = & 0 \\ z^{(iv)} - z & = & = \end{array}\right\}$$

(1) (2) (1) into (2)

Thus

$y = Ae^x + Be^{-x} + C\cos x + D\sin x$

$z = Ae^x + Be^{-x} - C\cos x - D\sin x$

Boundary conditions:

$y(0) = z(0) = 0 \Rightarrow A + B = 0,\ C = 0$

$y\left(\frac{\pi}{2}\right) = z\left(\frac{\pi}{2}\right) = 1 \Rightarrow A^{-1} = 2\sinh\frac{\pi}{2},\ D = 0$

$\Rightarrow y = \frac{\sinh x}{\sinh\frac{\pi}{2}},\ z = \frac{\sinh x}{\sinh\frac{\pi}{2}}$

NB could probably guess similarity of solution from symmetry of f in y, y' and z, z'.