In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

Find the continuous curve between the points $(0,0)$ and $(\frac{\pi}{2}, 1)$ for which

$$I = \int_0^{\frac{\pi}{2}} dxy(1-y'^2)^{\frac{1}{2}}$$

is stationary.

Answer

$F = y\sqrt{1-y'^2}$ Again $F = F(y, y')$ only so

$y'\dfrac{\partial F}{\partial y} - F = const$

$\dfrac{\partial F}{\partial y'} = \dfrac{y}{\sqrt{1-y'^2}} \times -\dfrac{2y'}{2}$

Therefore $-\dfrac{y'^2 y}{\sqrt{1-y'^2}} - y\sqrt{1-y'^2} = const$

$\Rightarrow \dfrac{y}{\sqrt{1-y'^2}} = const = \alpha$ say

$\Rightarrow y'^1 2 = 1 - \dfrac{y^2}{\alpha^2}$

Hence $\displaystyle\int \frac{dy}{\sqrt{1-\frac{y^2}{\alpha^2}}} = \int dx \Rightarrow y = \alpha \sin\left(\frac{x}{\alpha} + c\right)$

(standard integral) c=constant of integration

Now satisfy boundary conditions: $\left\{\begin{array}{rcl} y(0) = 0 & \Rightarrow & c = 0 \\ y\left(\dfrac{\pi}{2}\right) = 1 & \Rightarrow & 1 = \alpha\sin\left(\dfrac{\pi}{2\alpha}\right) \end{array}\right.$

Obvious solutions to the equation are $\alpha = \pm 1$. These are the only two (sensible) ones. Why?

Plot $\dfrac{1}{\alpha}$ and $\sin\dfrac{\pi}{2\alpha}$ against $\dfrac{1}{\alpha}$:

PICTURE

$\frac{1}{\alpha} = 0$ is also a solution but is meaningless as $\Rightarrow \alpha = \infty$, so a nonsensical solution.

Hence $y = \pm\sin(\pm x) \Rightarrow \underline{y = \sin x \text{uniquely}}$