In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

Find the continuous curve $y = y(x)$ between the points $(-1, 0)$ and $(1, 0)$ for which

$$I = \int_{-1}^{1} dx\sqrt{(1-y)}\sqrt{1+y'^2}$$

is stationary.

Answer

Since $F = \sqrt{1-y}\sqrt{1+y'^2}$ is not a function of x, an integral of the E-L equation is:

$y'\dfrac{\partial F}{\partial y'} - F = const.$

$\Rightarrow \dfrac{\partial F}{\partial y'} = \dfrac{\sqrt{1-y}}{\sqrt{1+y'^2}} \times \dfrac{2y'}{2}$

Thus $\dfrac{y'^2\sqrt{1-y}}{\sqrt{1+y'^2}} - \sqrt{1-y}\sqrt{1+y'^2} = const$

$\Rightarrow \sqrt{\dfrac{1-y}{1+y'^2}} = const = \alpha$ say

Therefore $y' = \pm\left(\dfrac{1}{\alpha^2} - 1 - \dfrac{y}{\alpha^2}\right)^{\frac{1}{2}}$

Therefore $x + c = \mp 2\alpha^2\left(\dfrac{1}{\alpha^2} - 1 - \dfrac{y}{\alpha^2}\right)^{\frac{1}{2}}$ (standard integration)

or $(x+c)^2 = 4\alpha^2\left(\dfrac{1}{\alpha}^2 - 1 - \dfrac{y}{\alpha^2}\right)$

But $y = 0$ at $x = \pm 1 \Rightarrow C = 0,\ \alpha^2 = \dfrac{1}{2}$ and so $\underline{y = \dfrac{1}{2}(1-x^2)}$