In what follows you may assume that the following notation applies

$$y = y(x),\ y' = \frac{dy}{dx}.$$

You may also assume that, unless otherwise stated, y is a sufficiently continuously differentiable function.

Question

Find the curves $y = y(x)$ which make the following functional integrals stationary

(i) $\int_0^1 y'\,dx$

(ii) $\int_0^1 yy'\,dx$

(iii) $\int_0^1 xyy'\,dx$

where is each case y satisfies the boundary conditions $y(0) = 0, y(1) = 1$.

Answer

(i) $F(y, y', x) = y'$:

EL equation is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

Thus stationary integral occurs when

$\frac{\partial F}{\partial y} = 0;\ \frac{\partial F}{\partial y'} = 1 \Rightarrow 0 - \frac{d}{dx}(1) = 0$

i.e., $0 = 0$

<u>i.e., for any $y(x)$ you like!</u>

[Why?

$\int_0^1 y'\,dx = \int_0^1 \frac{dy}{dx}\,dx = \int_{y(0)}^{y(1)} dy = |ds \int_0^1 dy = 1$

for <u>all</u> $y = y(x)$. Same answer so always stationary.]

(ii) $F(y, y', x) = y'$

$\dfrac{\partial F}{\partial y} = y'; \ \dfrac{\partial F}{\partial y'} = y$

so E-L equation becomes

$$\begin{aligned} y' - \frac{d}{dx}(y) &= 0 \\ \Rightarrow y' - y' &= 0 \\ 0 &= 0 \end{aligned}$$

$\Rightarrow$ stationary integral for any $y(x)$ you like (as above)

[Why?

$$\begin{aligned} \int_0^1 y'y\,dx &= \int_0^1 \frac{d}{dx}\left(\frac{y^2}{2}\right) dx \\ &= \left[\frac{y^2}{2}\right]_{y(0)}^{y(1)} \\ &= \left[\frac{y^2}{2}\right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

Same value for whatever y you pick, so stationary!]

(iii) $F(y, y', x) = xyy'$

$\dfrac{\partial F}{\partial y} = xy'; \ \dfrac{\partial F}{\partial y'} = xy$ so E-L equation becomes

$$\begin{aligned} xy' - \frac{d}{dx}(xy) &= 0 \\ \Rightarrow \quad xy' - y - xy' &= 0 \\ \Rightarrow \quad y &= 0 \end{aligned}$$

So apparently $y = 0$ for all x is the curve which minimises the integral. Indeed it satisfies the E-L equation, but <u>NOT</u> the boundary condition that $y(1) = 1$. Consequently there is <u>NO</u> continuous curve which makes $\int_0^1 xyy'\,dx$ stationary <u>and</u> satisfies the boundary conditions.