Question

The equation for the radial displacement $u(r,t)$ of a vibrating circular membrane of radius ℓ is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

together with the boundary condition $u(\ell,t) = 0$. If the membrane satisfies the initial conditions

$$u(r,0) = 0, \qquad \frac{\partial u}{\partial r}(r,0) = v_0$$

derive the solution

$$u(r,t) = \frac{2\ell v_0}{c}\sum_{n=1}^{\infty}\frac{J_0(\gamma_n r/\ell)\sin(\gamma_n ct/\ell)}{\gamma_n^2 J_1(\gamma_n)}$$

where γ_n is the n-th positive root of $J_0(x)$.

Answer

$$\frac{X''}{X} = \frac{1}{k}\frac{T''}{T} = \lambda$$
$$\Rightarrow X'' - \lambda X = 0$$

$X(0) = 0$ and $X(l) = 0$

$$\Rightarrow \lambda = -\frac{n^2\pi^2}{l^2}$$

Therefore

$$\begin{aligned} x_n(x) &= A_n \sin\left(\frac{n\pi x}{l}\right) \\ T' &= -\frac{n^2\pi^2 k}{l^2}T \\ T_n(t) &= B_n e^{-\frac{n^2\pi^2 kt}{l^2}} \end{aligned}$$

Thus

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2 kt}{l^2}} \\ f(x) &= \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

If $f(x) = 3\sin\left(\frac{4\pi x}{l}\right)$ then $Q_4 = 3$ and $Q_n = 0$ otherwise. So

$$u(x,t) = 3\sin\left(\frac{4\pi x}{l}\right)e^{-\frac{16\pi^2 kt}{l^2}}$$

If $l = 1$ then $f(x) = \sum_{n-1}^{\infty} Q_n \sin(n\pi x)$. But $f(x) = \sin(2\pi x) - 2\sin(5\pi x)$
$\Rightarrow Q_2 = 1 \;\; Q_5 = -2$ and $Q_n = 0$ otherwise. So

$$u(x,t) = \sin(2\pi x)e^{-400\pi^2 t} - 2\sin(5\pi x)e^{-2500\pi^2 t}$$