Question

The transverse displacement $y(x,t)$ of a vibrating string of length l, fixed at its endpoints, satisfies the equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 t}{\partial t^2} = 0.$$

(a) If the string is initially at rest with displacement $y(x,0) = f(x)$, where

$$f(x) = \begin{cases} x, & 0 \leq x < l/4 \\ l/4, & l/4 \leq x < 3l/4 \\ l-x, & 3l/4 \leq x \leq l \end{cases}$$

show that the displacement at later times is

$$y(x,t) = \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

with

$$Q_n = \frac{4l}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)$$

(b) Find the displacement if instead the string is initially in its equilibrium position, so that $y(x,0) = 0$, but has non-zero initial velocity $y_t(x,0) = g(x)$ given by

$$g(x) = \begin{cases} x, & 0 \leq x \leq l/2 \\ l-x, & l/1 < x \leq l \end{cases}$$

Answer

(a) Let $y(x,t) = X(x)T(t)$, so

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda$$

$$\Rightarrow X'' - \lambda X = 0$$

Using $X(0) = 0$ and $X(l) = 0 \Rightarrow \lambda = -\dfrac{n^2\pi^2}{l^2}$

Thus

$$\begin{aligned}X_n(x) &= A_n \sin\left(\frac{n\pi x}{l}\right)\\ T_n(t) &= B_n \sin\left(\frac{n\pi ct}{l}\right) + C_n \cos\left(\frac{n\pi ct}{l}\right)\end{aligned}$$

Using $T'(0) = 0 \Rightarrow B_n = 0$

So

$$\begin{aligned}y_n(x,t) &= Q_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\\ y(x,t) &= \sum Q_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\end{aligned}$$

and

$$y(x,0) = f(x) - \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right)$$

$$Q_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx$$

$$\begin{aligned}\frac{Q_n}{2} &= \int_0^{\frac{l}{4}} x\sin\left(\frac{n\pi x}{l}\right)\, dx + \frac{l}{4}\int_{\frac{l}{4}}^{\frac{3l}{4}} \sin\left(\frac{n\pi x}{l}\right)\, dx\\ &\quad + \int_{\frac{3l}{4}}^{l} (l-x)\sin\left(\frac{n\pi x}{l}\right)\, dx\\ \\ &= \left[-\frac{lx}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{4}} + \int_0^{\frac{l}{4}} \frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\, dx\\ &\quad + \left[-\frac{l^2}{4n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_{\frac{l}{4}}^{\frac{3l}{4}} + \left[\frac{-(l-x)}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_{\frac{3l}{4}}^{l}\\ &\quad - \int_{\frac{3l}{4}}^{l} \frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right) dx\\ &= -\frac{l^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right) + \left[\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{4}}\\ &\quad - \frac{l^2}{4n\pi}\cos\left(\frac{3n\pi}{4}\right) + \frac{l^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right)\\ &\quad + \frac{l^2}{4n\pi}\cos\left(\frac{3n\pi}{4}\right) - \left[\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_{\frac{3l}{4}}^{l}\end{aligned}$$

$$= \frac{l^2}{n^2\pi^2}\left\{\sin\left(\frac{n\pi}{4}\right)+\sin\left(\frac{3n\pi}{4}\right)\right\}$$

$$\frac{lQ_n}{2} = \frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)\cos\left(\frac{3n\pi}{4}\right)$$

(b) As Before: $y(x,t) = X(x)T(t)$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda$$
$$\Rightarrow X'' - \lambda X = 0$$

Using $X(0) = 0$ and $X(l) = 0 \Rightarrow \lambda = -\frac{n^2\pi^2}{l^2}$

Thus

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{l}\right)$$
$$T_n(t) = B_n \sin\left(\frac{n\pi ct}{l}\right) + C_n \cos\left(\frac{n\pi ct}{l}\right)$$

Using $y(x,0) = 0 \Rightarrow T(0) = 0 \Rightarrow C_n = 0$

So

$$y_n(x,t) = Q_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$
$$y(x,t) = \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} Q_n \sin\left(\frac{n\pi x}{l}\right) - \frac{l}{n\pi c}\cos\left(\frac{n\pi ct}{l}\right)$$
$$\text{so } \ g(x) = \sum_{n=1}^{\infty} -\frac{l}{n\pi c}Q_n \sin\left(\frac{n\pi x}{l}\right)$$
$$-\frac{lQ_n}{n\pi c} = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$-\frac{l^2Q_n}{2n\pi c} = \int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$= \int_0^{\frac{l}{2}} x\sin\left(\frac{n\pi x}{l}\right)dx + \int_{\frac{l}{2}}^l (l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$= \left[-\frac{lx}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{2}} + \int_0^{\frac{l}{2}}\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)dx$$

$$
\begin{aligned}
& + \left[-\frac{(l-x)}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \right]_{\frac{l}{2}}^{l} - \int_{\frac{l}{2}}^{l} \frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) dx \\
= \; & -\frac{l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[\frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right) \right]_0^{\frac{l}{2}} \\
& + \frac{l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right) - \left[\frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{l}\right) \right]_{\frac{l}{2}}^{l} \\
= \; & \frac{l^2}{n^2\pi^2} \left(2 \sin\left(\frac{n\pi}{2}\right) \right) \\
= \; & \begin{cases} 0 & n \;\; \text{even} \\ \dfrac{2l^2}{(2k+1)^2\pi^2} \sin\left((2k+1)\dfrac{\pi}{2}\right) & n \;\; \text{odd} \end{cases} \\
-\frac{l^2 Q_n}{2n\pi c} = \; & \begin{cases} 0 & n = 2k \\ \dfrac{2(-1)^k l^2}{(2k+1)^2\pi^2} \sin\left((2k+1)\dfrac{\pi}{2}\right) & n = 2k+1 \end{cases} \\
\Rightarrow Q_n = \; & \begin{cases} 0 & n = 2k \\ \dfrac{4c(-1)^{k+1}}{(2k+1)\pi} \sin\left((2k+1)\dfrac{\pi}{2}\right) & n = 2k+1 \end{cases} \\
\Rightarrow y(x,t) = \; & \sum_{k=0}^{\infty} \frac{4c(-1)^k}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi x}{l}\right) \sin\left(\frac{(2k+1)\pi c t}{l}\right)
\end{aligned}
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