Vector Calculus
Grad, Div and Curl

Question

$\underline{F}$ is a 2-dimensional smooth vector field.
C_ϵ is a circle of radius ϵ centred at the origin.
$\hat{\underline{N}}$ is the unit outward normal to C_ϵ.
Show that

$$\lim_{\epsilon \to O^+} \frac{1}{\pi\epsilon^2} \oint_{C_\epsilon} \underline{F} \bullet \hat{\underline{N}}\, ds = \mathrm{div}\underline{F}(0,0)$$

Answer

Use the Maclaurin expansion of $\underline{F}$,

$$\underline{F} = \underline{F}_0 + \underline{F}_1 x + \underline{F}_2 y + \cdots$$

with

$$\begin{aligned}
\underline{F}_0 &= \underline{F}(0,0) \\
\underline{F}_1 &= \frac{\partial}{\partial x}\, \underline{F}(x,y)\big|_{(0,0)} = \left.\left(\frac{\partial F_1}{\partial x}\underline{i} + \frac{\partial F_2}{\partial x}\underline{j}\right)\right|_{(0,0)} \\
\underline{F}_2 &= \frac{\partial}{\partial y}\, \underline{F}(x,y)\big|_{(0,0)} = \left.\left(\frac{\partial F_1}{\partial 2}\underline{i} + \frac{\partial F_2}{\partial 2}\underline{j}\right)\right|_{(0,0)}
\end{aligned}$$

Here, $\cdots$ represent terms in x and y of degree 2 and higher.
On the curve C_ϵ of radius ϵ centered at the origin, $\hat{\underline{N}} = \frac{1}{\epsilon}(x\underline{i} + y\underline{j})$.
$\Rightarrow$

$$\begin{aligned}
\underline{F} \bullet \hat{\underline{N}} &= \frac{1}{\epsilon}(\underline{F}_0 \bullet \underline{i}x + \underline{F}_0 \bullet \underline{j}y + \underline{F}_1 \bullet \underline{i}x^2 \\
&\quad + \underline{F}_1 \bullet \underline{j}xy + \underline{F}_2 \bullet \underline{i}xy + \underline{F}_2 \bullet \underline{j}y^2 + \cdots
\end{aligned}$$

Here $\cdots$ represents terms in x and y of degree 3 or higher.
Since

$$\begin{aligned}
\oint_{C_\epsilon} x\, ds &= \oint_{C_\epsilon} y\, ds = \oint_{C_\epsilon} xy\, ds = 0 \\
\oint_{C_\epsilon} x^2\, ds &= \oint_{C_\epsilon} y^2\, ds = \int_0^{2\pi} \epsilon^2 \cos^2\theta \epsilon\, d\theta = \pi\epsilon^3
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{1}{\pi\epsilon^2} \oint_{C_\epsilon} \underline{F} \bullet \hat{\underline{N}}\, ds &= \frac{1}{\pi\epsilon^2}\frac{\pi\epsilon^3}{\epsilon}(\underline{F}_1 \bullet \underline{i} + \underline{F}_2 \bullet \underline{j}) + \cdots \\
&= \mathrm{div}\underline{F}(0,0) + \cdots
\end{aligned}$$

Here $\cdots$ represents terms in ϵ of degree 1 or higher.
So taking the limit as $\epsilon \to 0$ gives

$$\lim_{\epsilon \to O^+} \frac{1}{\pi \epsilon^2} \oint_{C_\epsilon} \underline{F} \bullet \underline{\hat{N}} \, ds = \mathrm{div} \underline{F}(0,0)$$