Vector Calculus
Grad, Div and Curl

Question
$\underline{F}$ is a 3-dimensional smooth vector field.
$B_{a,b,c}$ is the surface of the box defined by

$$\begin{array}{rcl} -a \leq & x & \leq a \\ -b \leq & y & \leq b \\ -c \leq & z & \leq c \end{array}$$

with outward normal $\hat{\underline{N}}$.
Show that

$$\lim_{a,b,c \to 0^+} \frac{1}{8abc} \oiint_{B_{a,b,c}} \underline{F} \bullet \hat{\underline{N}} \, dS = \underline{\nabla} \bullet \underline{F}(0,0,0)$$

Answer
Use the Maclaurin expansion of $\underline{F}$:

$$\underline{F} = \underline{F}_0 + \underline{F}_1 x + \underline{F}_2 y + \underline{F}_3 z + \cdots$$

with

$$\begin{aligned}
\underline{F}_0 &= \underline{F}(0,0,0) \\
\underline{F}_1 &= \left.\frac{\partial}{\partial x}\underline{F}(x,y,z)\right|_{(0,0,0)} = \left.\left(\frac{\partial F_1}{\partial x}\underline{i} + \frac{\partial F_2}{\partial x}\underline{j} + \frac{\partial F_3}{\partial x}\underline{k}\right)\right|_{(0,0,0)} \\
\underline{F}_2 &= \left.\frac{\partial}{\partial y}\underline{F}(x,y,z)\right|_{(0,0,0)} = \left.\left(\frac{\partial F_1}{\partial y}\underline{i} + \frac{\partial F_2}{\partial y}\underline{j} + \frac{\partial F_3}{\partial y}\underline{k}\right)\right|_{(0,0,0)} \\
\underline{F}_3 &= \left.\frac{\partial}{\partial z}\underline{F}(x,y,z)\right|_{(0,0,0)} = \left.\left(\frac{\partial F_1}{\partial z}\underline{i} + \frac{\partial F_2}{\partial z}\underline{j} + \frac{\partial F_3}{\partial z}\underline{k}\right)\right|_{(0,0,0)}
\end{aligned}$$

$\cdots$ represents terms in x, y and z that are of degree two or above.
On the top of the box: $z = c$, $\hat{\underline{N}} = \underline{k}$.
On the bottom of the box: $z = -c$, $\hat{\underline{N}} = -\underline{k}$
On both of these: $dS = dxdy$
So

$$\begin{aligned}
& \left(\iint_{top} + \iint_{top}\right) \underline{F} \bullet \hat{\underline{N}} dS \\
= & \int_{-a}^{a} dx \int_{-b}^{b} dy (c\underline{F}_3 \bullet \underline{k} - c\underline{F} \bullet (-\underline{k} + \cdots \\
= & \; 8abc\underline{F}_3 \bullet \underline{k} + \cdots + 8abc \left.\frac{\partial}{\partial z} F_3(x,y,z)\right|_{0,0,0} + \cdots
\end{aligned}$$

Here, $\cdots$ represented terms in a, b and c that are of degree 4 or higher.
Similar formulas can be used for the other two face pairs.
Combining the three formulas gives

$$\oint\!\!\!\oint_{B_{a,b,c}} \underline{F} \bullet \underline{\hat{N}}\, dS = 8abc\,\mathrm{div}\underline{F}(0,0,0) + \cdots$$

So it can be seen that

$$\lim_{a,b,c\to 0^+} \frac{1}{8abc} \oint\!\!\!\oint_{B_{a,b,c}} \underline{F} \bullet \underline{\hat{N}}\, dS = \mathrm{div}\underline{F}(0,0,0)$$