Vector Calculus
Grad, Div and Curl

Question
Calculate **divF** and **curlF** for the vector field
$\underline{F}(r, \theta) = r\underline{i} + \sin\theta \underline{j}$
Given that (r, θ) are polar coordinates in the plane.
Answer
Since $x = r\cos\theta$ and $y = r\sin\theta$, we have $r^2 = x^2 + y^2$.
So

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta \\
\frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta \\
\frac{\partial}{\partial x}\sin\theta &= \frac{\partial}{\partial x}\frac{y}{r} = -\frac{xy}{r^3} \\
&= -\frac{\cos\theta\sin\theta}{r} \\
\frac{\partial}{\partial y}\sin\theta &= \frac{\partial}{\partial y}\frac{y}{r} = \frac{1}{r} - \frac{y^2}{r^3} = \frac{x^2}{t^3} \\
&= \frac{\cos^2\theta}{r} \\
\frac{\partial}{\partial x}\cos\theta &= \frac{partial}{\partial x}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3} \\
&= \frac{\sin^2\theta}{r} \\
\frac{\partial}{\partial y}\cos\theta &= \frac{\partial}{\partial y}\frac{x}{r} = -\frac{xy}{r^3} \\
&= -\frac{\cos\theta\sin\theta}{r}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\text{div}\underline{F} &= \frac{\partial r}{\partial x} + \frac{\partial}{\partial y}\sin\theta = \cos\theta + \frac{\cos^2\theta}{r} \\
\text{curl}\underline{F} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r & \sin\theta & 0 \end{vmatrix} \\
&= \left(-\frac{\sin\theta\cos\theta}{r} - \sin\theta\right)\underline{k}
\end{aligned}$$