Question

For each of the given functions, calculate its Taylor series about the given point; also, determine the radius and interval of convergence of the resulting power series whereever possible.

1. $f(x) = x^3 + 6x^2 + 5x - 7$ about $a = 6$;
2. $f(x) = e^{3x}$ about $a = -2$;
3. $f(x) = \cosh(x)$ about $a = 1$;

Answer

1. we start by calculating the derivatives of f at $a = 6$:

   $f^{(0)}(6) = f(6) = 455$; $f^{(1)}(6) = f'(6) = 185$; $f^{(2)}(6) = 48$; $f^{(3)}(6) = 6$; $f^{(n)}(6) = 0$ for $n \geq 4$.

   Hence, the Taylor series for f centered at $a = 6$ is

   $$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(6)(x-6)^n = 455 + 185(x-6) + \frac{1}{2}\,48(x-6)^2 + \frac{1}{6}\,6(x-6)^3.$$

   The radius of convergence of this series is ∞ (using the root test, for instance), and so the interval of convergence is $\mathbf{R}$.

2. we start by calculating that $f^{(n)}(x) = 3^n e^{3x}$ for $n \geq 0$, and so $f^{(n)}(-2) = 3^n e^{-6}$. Hence, the Taylor series for f centered at $a = -2$ is

   $$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(-2)(x+2)^n = e^{-6} \sum_{n=0}^{\infty} \frac{3^n}{n!}(x+2)^n.$$

   The radius of convergence of this series is ∞ (using the ratio test, for instance), and so the interval of convergence is $\mathbf{R}$.

3. we start here by recalling that

   $$f^{(n)}(x) = \begin{cases} \cosh(x) & \text{for } x \text{ even, and} \\ \sinh(x) & \text{for } x \text{ odd.} \end{cases}$$

   So, we have that $f^{(n)}(1) = \cosh(1) = \frac{1}{2}(e + \frac{1}{e})$ for n even, and $f^{(n)}(1) = \sinh(1) = \frac{1}{2}(e - \frac{1}{e})$ for n odd. Hence, the Taylor series for f centered at $a = 1$ is

   $$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(1)(x-1)^n &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} f^{(2k)}(1)(x-1)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} f^{(2k+1)}(1)(x-1)^{2k+1} \\ &= \frac{e^2+1}{2e} \sum_{k=0}^{\infty} \frac{1}{(2k)!}(x-1)^{2k} + \frac{e^2-1}{2e} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(x-1)^{2k+1}. \end{aligned}$$

   The radius of convergence of this series is ∞ (using the ratio test, for instance), and so the interval of convergence is $\mathbf{R}$.