Question
A Markov chain consists of a simple random walk taking place on a circle. The states consist of equally spaced points labelled $0,\ 1,\ 2,\ \cdots,\ a-1$ in a clockwise direction. At each step of the random walk transition takes place as follows:

(i) a clockwise step with probability p,

(ii) an anticlockwise step with probability q,

(iii) no change of position with probability $1-p-q$,

where $pq \neq 0$, and $p+q<1$.

Write down the transition matrix of the Markov chain.

Explain how the classification theorems enable you to deduce that in this case there is a long term equilibrium state occupancy distribution. Find this distribution. Find the mean recurrence time for any positive recurrent states.

Answer
PICTURE
The transition matrix is as follows

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \\ a-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & & & a-1 \end{array} \\ \begin{pmatrix} 1-p-q & p & 0 & \cdots & 0 & q \\ q & 1-p-q & p & & & 0 \\ 0 & q & 1-p-q & p & & 0 \\ \vdots & & & & & \vdots \\ 0 & & & q & 1-p-q & p \\ p & 0 & \cdots & 0 & q & 1-p-q \end{pmatrix} \end{array}$$

Now all the states intercommunicate, so they are of the same type (positive recurrent). Since $1-p-q \neq 0$ the states are all aperiodic. Thus we have a finite irreducible aperiodic distinction, which is also the equilibrium distribution.
Let $\boldsymbol{\pi}= (\pi_0, \pi_1, ..., \pi_{a-1})$ denote the stationary distribution. If it satisfies $\boldsymbol{\pi}$ $= \boldsymbol{\pi}$P. Thus we have

$$\begin{aligned} \pi + 0 &= (1-p-q)\pi_0 + q\pi_1 + p\pi_{a-1} \\ \pi_k &= p\pi_{k-1} + (1-p-q)\pi_k + q\pi_{k+1} \\ \pi_{a-1} &= q\pi_0 + p\pi_{a-2} - (1-p-q)\pi_{a-1} \end{aligned}$$

Note Columns sum to 1. Therefore (1,1,...,1) is a fixed vector and therefore $\boldsymbol{\pi}= \left(\frac{1}{a}, \frac{1}{a}, ..., \frac{1}{a}\right)$

$$
\begin{aligned}
p\pi_{a-1} + q\pi_1 - (p+q)\pi_0 &= 0 \qquad (1)\\
q\pi_{k+1} - (p+q)\pi_k + p\pi_{k-1} &= 0 \qquad (2)\\
q\pi_0 + p\pi_{a-2} - (p-q)\pi_{a-1} &= 0 \qquad (3)
\end{aligned}
$$

Solving (2) gives

$$\pi_k = A + b\left(\frac{p}{q}\right)^k \qquad if\, p \neq q$$

$$\pi_k = A + Bk \qquad if\, p = q$$

<u>Case 1</u> $p \neq q$. From (1) we have

$$
\begin{aligned}
p\left(A + B\left(\frac{p}{q}\right)^{a-1}\right) + q\left(A + B\left(\frac{p}{q}\right)\right) & \\
-(p+q)\left(A + B\left(\frac{p}{q}\right)^{a-1}\right) &= 0\\
\text{i.e. } B\left[p\left(\frac{p}{q}\right)^{a-1} - q\right] &= 0 \qquad (4)
\end{aligned}
$$

So either B = 0 or $\left[p\left(\frac{p}{q}\right)^{a-1} - q\right] = 0.$

From (3)

$$
\begin{aligned}
q(a+b) + p\left(A + b\left(\frac{p}{q}\right)^{a-2}\right) - (p+q)\left(a + b\left(\frac{p}{q}\right)^{a-1}\right) &= 0\\
B\left(\left\{p\left(\left(\frac{p}{q}\right)^{a-2} - \left(\frac{p}{q}\right)^{a-1}\right)\right\} - \left[p\left(\frac{p}{q}\right)^{a-1} - q\right]\right) &= 0 \qquad (5)
\end{aligned}
$$

From (4) if [ ] =0 , since {} $\neq 0$ in (5), we deduce B=0.
Thus $B = 0$ and $\pi_k = A$
<u>Case 2</u> $p = q$
From(1) we have

$$P(A + B(a-1)) + q(A+B) - (p+q)A = 0$$

i.e. $B[p(a-1)+q]=0$. Now $\quad > 0$ so B=0 and again $\pi_k = A$.
Now $\sum_{k=0}^{a-1} \pi_k = 1$ so $A = \frac{1}{a}$ and so $\boldsymbol{\pi}=\left(\frac{1}{a}, \frac{1}{a}, ..., \frac{1}{a}\right)$
The mean recurrence times are the reciprocals of the equilibrium probabilities. i.e. all equal to a.