Question

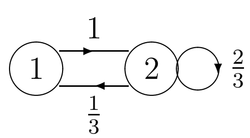
(a) A Markov chain has the infinite transition probability matrix given below. Classify the states, justifying your conclusions. Find the mean recurrence time for any positive recurrent states. (Label the states $1,\ 2,\ 3,\ \cdots$ in order.)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{5}{7} & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & \frac{2}{9} & 0 & 0 & 0 & \frac{7}{9} & 0 & . & . \\ 0 & 0 & 0 & 0 & \frac{2}{11} & 0 & 0 & 0 & 0 & \frac{9}{11} & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

(b) Draw a transition diagram and write down the transition matrix for a Markov chain having four intercommunicating positive recurrent states each of period 3, and three intercommunicating transient states.

Answer

(a) $\{1, 2\}$ forms a closed irreducible finite subchain, so that both states are positive recurrent. $f_{22}^{(1)} = \dfrac{2}{3} > 0$, so that both states are aperiodic.



$$\begin{aligned} \mu_2 &= 1 \cdot \frac{2}{3} + 2 \cdot \frac{4}{3} = \frac{4}{3} \\ \mu_1 &= 2 \cdot \frac{1}{3} + 3 \cdot \frac{2}{3} \cdot \frac{1}{3} + 4\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} + 5 \cdot \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} + ... \\ &= \frac{2}{3} + \frac{1}{3}\left(3 \cdot \frac{2}{3} + 4 \cdot \left(\frac{2}{3}\right)^2 + ...\right) \end{aligned}$$

either.

$$
\begin{aligned}
\text{Let } s &= 3\frac{2}{3} + 4\left(\frac{2}{3}\right)^2 + 5\left(\frac{2}{3}\right)^3 + \ldots \\
\frac{2}{3}s &= 3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^3 + \ldots \\
\frac{1}{3}s &= 2 + \left(\frac{2}{3}\right)^3 + \ldots \\
&= \frac{1}{3} + 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots \\
&= \frac{1}{3} + \frac{1}{1-\frac{2}{3}} = \frac{1}{3} + 3 \\
&= \frac{10}{3} \\
\text{So } \ \mu_1 &= \frac{2}{3} + \frac{1}{3} \cdot 10 = 4
\end{aligned}
$$

or.

$$
\frac{1}{\mu_1} + \frac{1}{\mu_2} = 1 \qquad \frac{1}{\mu_1} = \frac{1}{4}
$$

State 3 is transient, leading to $\{1, 2\}$ in one step.

State 4 is absorbing.

State $\{5, 6, 7, ...\}$ form a closed irreducible subchain. $f_{55}^{(1)} = \frac{1}{3} > 0$. so they are all aperiodic.

Consider state 5.

The Markov chain can return to 5 at the nth step ($n > 1$) only via the path

$$
5 \to 6 \to 7 \to \ldots \to 5 + (n-1) \to 5
$$

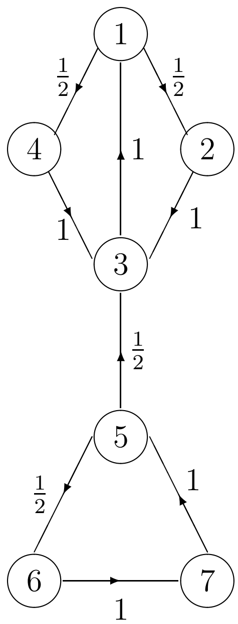
$$
\begin{aligned}
\text{So } \ f_{55}^{(n)} &= \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdots \frac{2n-3}{2n-1} \cdot \frac{2}{2n+1} = \frac{4}{(2n-1)(2n+1)} \\
&= \frac{2}{2n-1} - \frac{2}{2n+1} \\
\text{So } \ f_{55} &= \frac{1}{3} + \sum_{n=2}^{\infty} \left(\frac{2}{2n-1} - \frac{2}{2n+1} \right) = \frac{1}{3} + \frac{2}{3} = 1
\end{aligned}
$$

So state 5 is recurrent.

$$\mu_5 = \sum_{n=1}^{\infty} n f_{00}^{(n)} = \frac{1}{3} + \sum_{n=2}^{\infty} \frac{n}{(2n-1)(2n+1)} = \infty$$

Thus states 5, 6, 7... are all null-recurrent.

(b) Example



$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$