Question

A gambler with initial capital z plays against an infinitely rich opponent. At each play the gambler wins 2 with probability p, and loses 1 with probability $q = 1 - p$. Letting q_z denote the probability that the gambler will eventually be ruined, show that, for $z \geq 1$,

$$q_z = pq_{z+2} + qq_{z-1}$$

giving a careful explanation of your reasoning. What is the value of q_0?

Find the general solution of the above difference equation, and include a discussion of repeated roots when they occur. (The auxiliary equation has 1 as a root.)

Use the assumption that $q_z \to 0$ as $z \to \infty$, together with the value of q_0, to show that

$$q_z = 1 \text{ if q} \geq \text{2p},$$

$$q_z = \left[-\frac{1}{2} + \left(\frac{1}{4} + \frac{q}{p} \right)^{\frac{1}{2}} \right] \text{ if q} < \text{2p}.$$

Answer

we argue conditionally on the result of the first play, as follows:

$$\begin{aligned} q_z &= P(\text{gambler wins 1st bet and subsequently ruined}) \\ &+ P(\text{gambler loses 1st bet and subsequently ruined}) \\ \\ &= P(\text{ruin } | \text{wins 1st bet}) \cdot P(\text{wins 1st bet}) \\ &+ P(\text{ruin } | \text{loses 1st bet}) \cdot P(\text{loses 1st bet}) \\ \\ &= q_{z+2} \cdot p + q_{z-1} \cdot q \end{aligned}$$

Now $q_0 = 1$.

To solve the difference equation. The auxiliary equation is

$$p\lambda^3 - \lambda + q = 0$$

i.e.

$$(\lambda - 1)(p\lambda^2 + p\lambda - q) \quad (\text{using } p + q = 1)$$

So the roots are

$$\lambda_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{q}{p}} \qquad \lambda_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{q}{p}} \qquad \lambda_3 = 1$$

Now $\lambda_2 < -1$, and so the only possibilities for a repeated root is if $\lambda_1 = 1$ in that case

$$\sqrt{\frac{1}{4} + \frac{q}{p}} = \frac{3}{2} \quad \text{which gives} \quad q = 2p.$$

The general solution is

$$q_z = A + Bz + C\lambda_2^z$$

Since$\lambda_2 < -1$ this oscillates unboundedly if $c \neq 0$ and $\rightarrow \infty$ if $c = 0 \& B \neq 0$
Thus $q_z = A$ and so since $q_0 = 1$,

$$q_z = 1 \quad \text{for all } z$$

Now if $p \neq q \quad q_z = A + B\lambda_1^z + C\lambda_2^z$
Now $|\lambda_2| > |\lambda_1|$ and so if $C \neq 0$ q_z oscillates unboubdely as $z \rightarrow \infty$. So $C+0$.
If $q > 2p$ then $\lambda_1 > 1$ and so $q_z \rightarrow \pm\infty$ as $z \rightarrow \infty$ if $B \neq 0$. Thus A=0 by the given assumption and $q_0 = 1$ gives $B = 1$.

To summaries:
If

$$q \geq 2p \text{ then } \; q_z = 1$$

If

$$q < 2p \; q_z = \lambda_1^z = \left[-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{q}{p}}\right]^z$$