Question

Use the calculus of residues to show that

a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \frac{2\pi}{\sqrt{3}}$

b) $\displaystyle\sum_{r=-\infty}^{\infty} \frac{1}{\left(r+\frac{1}{2}\right)^2+1} = \pi\tanh\pi.$

Answer

a) Let $z = e^{i\theta}$ $\qquad d\theta = \dfrac{dz}{iz}$ $\qquad \sin\theta = \dfrac{1}{2i}\left(z-\dfrac{1}{z}\right)$ $\qquad C$ - the unit circle

$$I = \int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \int_C \frac{dz}{iz\left(2-\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)} = \int_C \frac{-2dz}{z^2-4iz-1}$$

The denominator has roots at $z = i(2\pm\sqrt{3})$, so the integrand has a simple pole at $z = i(2-\sqrt{3})$ inside C with residue

$$\frac{1}{i(2-\sqrt{3})-i(2+\sqrt{3})} = \frac{1}{-2i\sqrt{3}}$$

So $I = -2.2\pi i\dfrac{-1}{2i\sqrt{3}} = \dfrac{2\pi}{\sqrt{3}}$

b) Let $f(z) = \dfrac{1}{\left(z+\frac{1}{2}\right)^2+1} = \dfrac{1}{\left(z+\frac{1}{2}+i\right)\left(z+\frac{1}{2}-i\right)}$

This has simple poles at $-\dfrac{1}{2}-i$ and $-\dfrac{1}{2}+i$.

$$\mathrm{Res}\left(-\frac{1}{2}-i\right) = \frac{1}{-\frac{1}{2}-i+\frac{1}{2}-i} = -\frac{1}{2i}$$

$$\mathrm{Res}\left(-\frac{1}{2}+i\right) = \frac{1}{-\frac{1}{2}+i+\frac{1}{2}+i} = \frac{1}{2i}$$

Consider $\pi\cot\pi z f(z) = g(z)$

$$\mathrm{Res}\left(g, -\frac{1}{2}-i\right) = -\frac{\pi\cot\pi\left(-\frac{1}{2}-i\right)}{2i} = -\frac{\pi\tanh\pi}{2}$$

$$\operatorname{Res}\left(g, -\frac{1}{2}+i\right) = \frac{\pi \cot \pi \left(-\frac{1}{2}+i\right)}{2i} = -\frac{\pi \tanh \pi}{2}$$

Let Γ_N be the square with vertices $\pm\left(N+\dfrac{1}{2}\right)(1 \pm i)$ on Γ_N $\pi \cot \pi z$ is bounded uniformly and

$$\int_{\Gamma_N} g(z)dz \to 0 \text{ as } N \to \infty$$

$$\int_{\Gamma_N} = 2\pi i \left(\sum_{r=-N}^{N} f(r) - \pi \tanh \pi\right)$$

Letting $N \to \infty$ gives

$$\sum_{r=-\infty}^{\infty} \frac{1}{\left(r+\frac{1}{2}\right)^2 + 1} = \pi \tanh \pi$$