Question

a) The function $Q(z)$ is a rational function such that $\lim_{z\to\infty} zQ(z) = 0$, and the curve Γ is the upper half of the circle $|z| = R$. Prove that

$$\lim_{R\to\infty} \int_\Gamma Q(z)e^{\mathrm{im}z}dz = 0$$

where $m \geq 0$.

b) Use the result in part (a) and the calculus of residues to show that

i) $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$

ii) $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}.$

Answer

a) $Q(z)$ is a rational function and $\lim_{|z|\to\infty} zQ(z) = 0$.

Let $Q(z) = \dfrac{A(z)}{B(z)}$ where A and B are polynomials of degrees a and b.

$zQ(z) = \dfrac{zA(z)}{B(z)} \sim \dfrac{\text{degree } a+1}{\text{degree } b}$

so $a+1 < b$ since $zQ(z) \to 0$ as $|z| \to \infty$

i.e. $a \leq b-2$, so $\exists K, H$ such that for $|z| > H$ $\qquad |q(z)| < \frac{B}{|z|^2}$

Now for z on Γ $\qquad |e^{imz}| = e^{-mR\sin\theta} \leq 1$ for $\theta \in [0,\pi]$

So for $R > H$

$\left|\int_\Gamma Q(Z)e^{imz}dz\right| \leq \dfrac{K}{R^2}\pi R \to 0$ as $R \to \infty$

b) i) $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2}$

We integrate $f(z)\dfrac{1}{(1+z^2)^2}$ around C.

This satisfies the conditions of (a) with $m = 0$ and

$q(z) = \dfrac{1}{(1+z^2)^2}$

$f(z)$ has a pole of order 2 at $z = i$ inside C with residue

$$\frac{d}{dz}(z-i)^2 f(z)\bigg|_{z=i} = \frac{d}{dz}\frac{1}{(z+i)^2} = \frac{-2}{(z+i)^3}\bigg|_{z=i} = -\frac{i}{4}$$

$$\int_C f(z)dz = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$$

so $\displaystyle\int_0^\infty f(x)dx = \frac{\pi}{4}$

ii) We integrate $f(z) = \dfrac{e^{iz}}{1+z^2}$ around C, the result of (a) applies with $m = 1$ and $q(z) = \dfrac{1}{1+z^2}$

$f(z)$ has a simple pole at $z = i$ inside Γ with residue given by $\displaystyle\lim_{z\to i}\frac{e^i z}{z+i} = \frac{e^{-1}}{2i}$

So $\displaystyle\int_C \frac{e^{iz}}{1+z^2}dz = 2\pi i\frac{e^{-1}}{2i} = \frac{\pi}{e}$

So $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}$ and $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2} = \frac{\pi}{2e}$