Question

Show that the transformation $w = \dfrac{1+z}{1-z}$ maps the unit disc $|z| < 1$ conformally onto the half plane $u > 0$, where $w = u + iv$. What is the image of the upper half of the unit disc?

Show that the transformation $w = \exp z$, where $z = x + iy$, maps the half-strip $\Omega = \{(x, y) : 0 < y < \pi,\ x < 0\}$ conformally onto the upper half of the unit disc. how do the boundaries of Ω map?

Find a transformation which maps Ω conformally onto the first quadrant. Hence, or otherwise, obtain a solution $T(x, y)$ of Laplace's equation in Ω which satisfies the boundary conditions

$T(x, 0) = T(x, \pi) = 0;\ \ T(0, y) = 1.$

Answer

$w = \dfrac{1+z}{1-z} = \dfrac{1+e^{i\theta}}{1-e^{i\theta}} = \dfrac{e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} = \dfrac{\cos\frac{\theta}{2}}{-i\sin\frac{\theta}{2}} = i\cot\frac{\theta}{2}$

S0 $|z| = 1 \to$ imaginary axis.

Now $z = 0 \to w = 1$, so T maps D to U.

For the upper half of D, $0 < \theta < \pi$ on the boundary so $\cot\frac{\theta}{2} > 0$ thus the semicircle maps to the positive imaginary axis.

Also $-1 < z < 1$ real $\Rightarrow w > 0$ real.

So

DIAGRAM

$w = \exp z = e^x e^{iy}$

$y = 0 \Rightarrow w = e^x$ real and < 1 for $x < 0$

$y = \pi \Rightarrow w = -e^x$ real and > -1 for $x < 0$

$x = 0, \quad 0 < y < \pi \Rightarrow w = e^{iy}$

DIAGRAM

$w = \dfrac{1+e^z}{1-e^z}$ maps $\Omega \to$ first quadrant.