Question

a) Find the real and imaginary parts of the functions $\cos z$ and $\sin z$ where $z = x+iy$. Verify the Cauchy-Riemann equations in each case and show that $\frac{d}{dz}(\sin z) = \cos z$.

b) Prove that, unless z is a real number

$$|\sin z|^2 + |\cos z|^2 > 1.$$

c) Evaluate the integral $\int_C \tan z dz$ where C is the straight line segment from $z = 0$ to $z = \frac{\pi}{2}(1+i)$.

Express your answer in the form $a + ib$ where a and b are real.

Answer

a) $\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$

$$= \cos x \cosh y - i \sin x \sinh y$$

Similarly $\sin z = \sin x \cosh y + i \cos x \sinh y$

For $\cos z$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y \quad \frac{\partial v}{\partial y} = -\sin x \cosh y$$
$$\frac{\partial u}{\partial y} = \cos x \sinh y \quad -\frac{\partial v}{\partial x} = \cos x \sinh y$$

For $\sin z$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad \frac{\partial v}{\partial y} = \cos x \cosh y$$
$$\frac{\partial u}{\partial y} = \sin x \sinh y \quad -\frac{\partial v}{\partial x} = \sin x \sinh y$$

so the Cauchy-Riemann equations are satisfied, and the partial derivatives are all continuous.

$$\frac{d}{dz}(\sin z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos x \cosh y - i \sin x \sinh y = \cos z$$

b) $|\sin z|^2 + |\cos z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$\qquad + \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cosh^2 y + \sinh^2 y = \cosh 2y \geq 1$ with equality iff $y = 0$.

c) $C:\ z = (1+i)t \quad 0 \leq t \leq \frac{\pi}{2}$

$\int_C \tan z = \int_0^{\frac{\pi}{2}} \tan(1+i)t(1+i)dt = [-\log\cos(1+i)t]_0^{\frac{\pi}{2}}$

$= -\log\cos\left(\frac{\pi}{2} + i\frac{\pi}{2}\right) = -\log\left(-sin\left(i\frac{\pi}{2}\right)\right)$

$= -\log\left(-i\sinh\frac{\pi}{2}\right) = -\ln\sinh\frac{\pi}{2} + i\frac{\pi}{2}$