Question

A particle of mass m falls from rest due to gravity in a fluid which offers resistance force of size $k|v|^\beta$, where k and β are positive constants. Write down Newton's second law for the motion. Without solving the equation determine the limiting speed of the object in terms of the constants k and β. [Hint: if a finite limiting velocity exits then at that speed the resistance balances the gravitational force.] (note: beware of the direction of the forces) For the particular case $\beta = 2$ solve the equations to find the relation between the velocity and the time.

Answer

Falling particle $\Rightarrow v < 0$ and the air resistance acts upwards.

$$m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = -mg + k|v|^\beta$$

The limiting velocity is the same as the equilibrium point, it is when $-mg + k|v|^\beta = 0$

$$|v|^\beta = \frac{k}{mg} \Rightarrow |v| = \left(\frac{k}{mg}\right)^{\frac{1}{\beta}} = \text{speed.}$$

for $\beta = 2$, $\quad m\dfrac{dv}{dt} = -mg + kv^2$ this is separable and bernoulli.

$$\frac{dv}{dt} = -g + \frac{k}{m}v^2$$

$$\int \frac{dv}{\frac{k}{m}v^2 - g} = \int dt,$$ now use partial fractions, to get

$$\int \frac{\frac{1}{2\sqrt{g}}}{\left(\sqrt{\frac{k}{m}}v - \sqrt{g}\right)} + \frac{-\frac{1}{2\sqrt{g}}}{\left(\sqrt{\frac{k}{m}}v + \sqrt{g}\right)} dv = t + A$$

$$\frac{1}{2}\sqrt{\frac{m}{gk}} \ln\left(\frac{v - \sqrt{\frac{gm}{k}}}{v + \sqrt{\frac{gm}{k}}}\right) = t + A$$

$$v(t) = \left(\frac{1 + ce^{2\sqrt{\frac{gk}{m}}t}}{1 - ce^{2\sqrt{\frac{gk}{m}}t}}\right)\sqrt{\frac{gm}{k}}$$