Question

For larger distances from the earth the gravitational potential is more accurately represented by $-mMG/(x+R)$ where G is constant, M is the mass of the earth, m is the mass of the particle, x is the distance from the surface of the earth and R is the earth's radius. Show that the force produced by this potential at the surface of the earth is the same as the gravitational force $-mg$ if the constants are such that $g = MG/R^2$ Write down the total energy of a rocket of constant mass m travelling vertically. Assuming the rocket blasts off from the earth surface (where $x = 0$) with speed $v = V_0$ find the maximum height the rocket reaches. Determine the critical velocity that ensures the rocket never reaches a maximum height (this is the escape velocity).

Answer

Kinetic energy $= T = \frac{1}{2}mv^2$ Potential energy $= V = -\frac{mHG}{x+R}$

Energy $= T + V = \frac{1}{2}v^2 - \frac{mHG}{x+R}$

Force $= -\frac{dV}{dx} = -\frac{mHG}{(x+R)^2}$, near the earth's surface $x = c$

$\Rightarrow$ Force $\approx -\frac{\mathrm{mHG}}{\mathrm{R}^2}$. Compare this with $-mg$, and show these are the same if $g = \frac{HG}{R^2}$.

because the force only depends on x, it is conservative

$\Rightarrow$ energy remains constant.

Initially energy $= \frac{1}{2}mv_0^2 - \frac{mHG}{R}$

$\Rightarrow \frac{1}{2}mv^2 - \frac{mHG}{x+R} = \frac{1}{2}mv_0^2 - \frac{mHG}{R}$ always

so $v^2 = v_0^2 - \frac{2HG}{R} + \frac{2HG}{x+R}$

The maximum height occurs when $v = 0$

$\Rightarrow 0 = v_0^2 - \frac{2HG}{R} + \frac{2HG}{x+R}$

$\Rightarrow x = \frac{\frac{R^2v_0^2}{2HG}}{1 - \frac{Rv_0^2}{2HG}}$ is the maximum height.

Note:

for $1 - \frac{Rv_0^2}{2HG} > 0$ there is a maximum height that has $x > 0$.

for $1 - \frac{Rv_0^2}{2HG} < 0$ there is no maximum height that has $x > 0$.

$1 - \frac{Rv_0^2}{2HG} < 0 \Rightarrow 1 < \frac{Rv_0^2}{2HG}$

$\Rightarrow v_0 > \sqrt{\frac{2HG}{R}}$

when $v_0 = \sqrt{\frac{2HG}{R}}$ this is the escape velocity.