QUESTION

Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without evaluating the integral, show that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}.$$

ANSWER

Length of contour $= 4\pi/4 = \pi$. Also (looking at modulus of integrand)

$$\left| \frac{1}{z^2 - 1} \right| \leq \frac{1}{|z|^2 - 1} \leq \frac{1}{3}.$$

(We have used the backward triangle inequality here.) Thus by the Estimation Theorem

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \pi/3.$$