Question

The differential equation

$$y'' + \delta(xy' + y) = 0$$

(here δ is a constant) arises in the study of turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the other solution in the form of an integral. $(*)$

Answer

Check that $y_1 = e^x$ is a solution to the equation

This follows since $y_1' = e^x$ and $y_1'' = e^x$ and putting these into the equation makes it true.

Equation is in standard form with $p(x) = \dfrac{-x}{x-1}$ and $q(x) = \dfrac{1}{x-1}$. Use method of reduction of order with $y_2 = vy_1$ where $v(x) = \displaystyle\int \left(\frac{1}{y_1^2(x)}\, e^{-\int p(x)\,dx} \right) dx$

so that $v(x) = \displaystyle\int \left(\frac{1}{(e^x)^2}\, e^{-\int \frac{-x}{x-1}\,dx} \right) dx$

$\left\{ \text{Note that } \displaystyle\int \frac{x}{x-1}\,dx = \int \frac{x-1+1}{x-1}\,dx = \int 1 + \frac{1}{x-1} = x + \ln(x-1) \right\}$

$v(x) = \displaystyle\int e^{-2x}\, e^{x+\ln(x-1)}\,dx = \int e^{-x}(x-1)\,dx = \int xe^{-x}\,dx - \int e^{-x}\,dx$

$v(x) = \displaystyle\int xe^{-x}\,dx + e^{-x} = -xe^{-x} + \int e^{-x}\,dx + e^{-x} = -xe^{-x}$

Thus we have $y_2(x) = \left(-xe^{-x}\right) e^x = -x$

Hence the general solution is $y(x) = Ae^x + Bx$