Question

The Albanian singing flea is hermaphrodite, and at the beginning of the breeding season the population consists of b fleas. At any subsequent time during the breeding season an individual flea has, during any time interval of length δt, independent of previous history and other fleas,

(i) a probability $\alpha\delta t + o(\delta t)$ of producing one new flea,

(ii) a probability $\beta\delta t + o(\delta t)$ of producing two new fleas,

(iii) a probability $o(\delta t)$ of producing more then two new fleas,or dying, as $\delta t \to 0$

With the first winter frost a few fleas manage to hibernate to serve as the starting population for the next season, and the remainder perish.

Let$p_n(t)$ denote the probability the the total population size is n at time t (for values of t with in the breeding season). Show that for any n = 1, 2, ...

$$p_n'(t) = -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) + \beta(n-2)p_{n-2}(t).$$

Find the probability that, at time t, the population has not changed from its original from its original size.

Suppose that the mean number of fleas at time t is

$$M(t) = \sum_{n=0}^{\infty} np_n(t)$$

Show that $M'(t) = (\alpha + 2\beta)M(t)$ and hence find $M(t)$.

Answer

$$\begin{aligned} P(X(t+\delta t) = n+1|X(t) = n) &= \alpha n\delta t + o(\delta t) \\ P(X(t+\delta t) = n+2|X(t) = n) &= \beta n\delta t + o(\delta t) \\ P(X(t+\delta t) = n|X(t) = n) &= 1 - (\alpha+\beta)n\delta t + o(\delta t) \end{aligned}$$

Arguing conditionally gives:

$$\begin{aligned} p_n(t+\delta t) &= p_n(t)(1 - (\alpha+\beta)n\delta t + o(\delta t)) \\ &\quad + p_{n-1}(t)(\alpha(n-1)\delta t + o(\delta t)) \\ &\quad + p_{n-2}(t)(\beta(n-2)\delta t + o(\delta t)) \end{aligned}$$

(Note that for $k \leq b$ $p_k(t) = 0$, so we don't need special equations for $k \leq b + 2$)
Thus we have:

$$\begin{aligned}\frac{p_n(t+\delta t) - p_n(t)}{\delta t} &= -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) \\ &\quad +\beta(n-2)p_{n-2}(t) + o(\delta t)\end{aligned}$$

Letting $\delta t \to 0$ gives

$$p_n'(t) = -(\alpha+\beta)np_n(t) + \alpha(n-1)p_{n-1}(t) + \beta(n-2)p_{n-2}(t)$$

Now $p_b(0) = 1$ and $p_n(0) = 0$ for $n \neq b$
So $p_b'(t) = -(\alpha+\beta)bp_b(y)$ and $p_b(0) = 1$
Thus $p_b(t) = e^{-(\alpha+\beta)bt}$

Now

$$M(t) = \sum_{n=0}^{\infty} np_n(t) \qquad M'(t) = \sum_{n=0}^{\infty} np_n'(t)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} -n^2(\alpha+\beta)p_n(t) + \sum_{n=0}^{\infty} \alpha(n-1)np_{n-1}(t) + \sum_{n=0}^{\infty} \beta(n-2)np_{n-2}(t) \\
&= \sum_{n=0}^{\infty} -n^2(\alpha+\beta)p_n(t) + \sum_{n=0}^{\infty} \alpha(n+1)np_n(t) + \sum_{n=0}^{\infty} \beta(n+2)np_n(t) \\
&= \sum np_n(t)[-n^2(\alpha+\beta) + \alpha(n+1) + \beta(n+2)] \\
&= (\alpha+2\beta)M(t)
\end{aligned}$$

Now $M(0) = b$ and so

$$M(t) = be^{(\alpha+2\beta)t}$$