Question

A random walk has the infinite set {0, 1, 2, ...} as possible states. State 0 is a partially reflecting barrier. If state 0 is occupied at step n then states 0 and 1 are equally likely to be occupied at step n+1 of the random walk. For all other states, transitions of +1, -1, 0 take place with the probabilities p, q, $1-p-q$ respectively. Let $p_{j,k}^{(n)}$ denote the probability that the random walk is in state k at step n, having started in state j. Derive the difference equation

$$p_{j,k}^{(n)} = p \cdot p_{j,k-1}^{(n-1)} + q \cdot p_{j,k+1)}^{(n-1)} + (1-p-q) \cdot p_{j,k}^{(n-1)} \quad (k \geq 2)$$

giving clear explanation of the reasoning leading to the equation. Write down analogous equations for $k = 0$ and $k = 1$. The long-term equilibrium distribution is given by

$$\pi_k = \lim_{n\to\infty} p_{j,k}^{(n)} \qquad (j = 0, 1, 2, ...)$$

when these limits exist. Obtain a set of difference equations for (π_k). Solve these equations, recursively or otherwise, showing that if $p \geq q$ there is no solution, and finding explicit expressions for π_k in the case $p < q$. You may assume that $q \neq 0$.

Answer

Arguing conditionally on the last step gives

$$\begin{aligned} p_{jk}^{(n)} &= pp_{j,k-1}^{(n-1)} + qp_{j,k+1}^{(n-1)} + (1-p-q)p_{j,k}^{(n-1)} \\ p_{j0}^{(n)} &= \frac{1}{2}p_{j0}^{(n-1)} + qp_{j1}^{(n-1)} \\ p_{j1}^{(n)} &= \frac{1}{2}p_{j0}^{(n-1)} + qp_{j,2}^{(n-1)} + (1-p-q)p_{j,1}^{(n-1)} \end{aligned}$$

Taking limits as n $\to \infty$gives

$$\begin{aligned} \pi_k &= p\pi_{k-1} + q\pi_{k+1} + (1-p-q)\pi_k \quad k \geq 2 \\ \pi_0 &= \frac{1}{2}\pi_0 + q\pi_1 \\ \pi_1 &= \frac{1}{2}\pi_0 + q\pi_2 + (1-p-q)\pi_1 \end{aligned}$$

Rewriting these equations gives,

$$q\pi_1 = \frac{1}{2}\pi_0 \qquad (1)$$

$$q\pi_2 + (-p-q)\pi_1 + \frac{1}{2}\pi_0 = 0 \qquad (2)$$

$$q\pi_{k+1} + (-p-q)\pi_k + p\pi_{k-1} = 0 \qquad (3)$$

Using (1) in (2) gives $q\pi_2 = p\pi_1$ Assuming that $q\pi_k = p\pi_{k-1}$ gives, using (3) $q\pi_{k+1} = p\pi_k$
Hence by induction this is true for $k \geq 1$

Thus $\pi_k = \left(\frac{p}{q}\right)^{k-1} = \frac{1}{2q}\left(\frac{p}{q}\right)^{k-1} \pi_0 \quad k \geq 1$
Now for (π_k) to be a probability distribution, $\sum \pi_k = 1$
i.e. $\left(1 + \frac{1}{2q}\sum_{k=1}^{\infty}\left(\frac{p}{q}\right)^{k-1}\right) = 1$
If $p \geq q$ the series diverges, so there is no solution.
If $p < q$,

$$\pi_0\left(1 + \frac{1}{2q}\frac{1}{\left(1 - \frac{p}{q}\right)}\right) = 1; \quad \pi_0\left(1 + \frac{1}{2(q-p)}\right) = 1$$

So

$$\pi_0 = \frac{2(q-p)}{1+2(q-p)} \quad \text{and} \quad \pi_k = \frac{1}{2q}\left(\frac{p}{q}\right)^{k-1} \cdot \frac{2(q-p)}{1+2(q-p)} \quad (k \geq 1)$$