Question

A gambler with initial capital £z (where z is a positive integer) plays a coin tossing game against an infinity rich opponent. Two fair coins are tossed: if both show heads the gambler wins £2; if both show tails the gambler wins £1; otherwise the gambler loosed £1.

Letting q_z denote the probability the the gambler is eventuality ruined, show that

$$q_{z+2} + q_{z+1} - 4q_z + 2q_{z-1} = 0$$

Find the general solution of this difference equation. Using the assumption that $q_z \to 0$ as $4z \to \infty$, show that

$$q_z = (\sqrt{3} - 1)^z.$$

What is the minimum initial capital the gambler needs in order that he has a better than even chance of not being ruined?

Suppose the gambler starts with £1. Considering the various possible outcomes, show that the probability of ruin in 5 or fewer steps is $\frac{78}{128}$

Answer

Let q_z denote the probability of ruin, with initial capitial £z. Arguing conditionally on the result of the first bet gives

$$q_z = \frac{1}{4} \cdot q_{z+2} + \frac{1}{4} q_{z+1} + \frac{1}{2} q_{z-1}$$

Rearranging this gives

$$q_{z+2} + q_{z+1} - 4q_z + 2q_{z-1} = 0$$

To find the general solution put $q_z = \lambda^z$. This gives the auxillary equation

$$\begin{aligned} \lambda^3 + \lambda^2 - 4\lambda + 2 &= 0 \\ (\lambda - 1)(\lambda^+ 2\lambda - 2) &= 0 \\ \text{So } \lambda &= 1,\ -1 - \sqrt{3},\ -1 + \sqrt{3} \end{aligned}$$

Thus $q_z = A + B(-1 - \sqrt{3})^z + C(-1 + \sqrt{3})^z$
Now q_z is a probability and since $-1 - \sqrt{3} < -1$ and $0 < -1 + \sqrt{3} < 1$, we cannot have the righthand side between the) and 1 unless B=0. Now

$(-1-\sqrt{3})^z \to 0$ as $z \to \infty$ and so assuming $q_z \to 0$ as $z \to \infty$ we conclude that A=0. Hence

$$q_z = C(-1+\sqrt{3})^z,$$

and finally $q_0 = 1$ gives $C = 1$.

Now in order to have $q_z < \frac{1}{2}$ we must have $(-1+\sqrt{3})^z < \dfrac{1}{2}$ i.e., $z\ln(-1+\sqrt{3}) < -\ln 2$ giving $z > \dfrac{-\ln 2}{\ln(-1+\sqrt{3})} = 2.22...$ so £$z \geq$ £3.

Starting with £1, the paths leading to ruin in 5 or fewer steps are:

1 step	L	prob $\dfrac{1}{2}$
2 steps	IMPOSSIBLE	
3 steps	W(1) L L	$\dfrac{1}{2^4}$
4 steps	W(2) L L L	$\dfrac{1}{2^5}$
5 steps	W(1) W(1) L L L	$\dfrac{1}{2^7}$
	W(1) L W(1) L L	$\dfrac{1}{2^7}$

So $p = \dfrac{1}{2^7}(648411) = \dfrac{78}{128}$