QUESTION

(i) Evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ \beta & 5 & 1 \end{pmatrix},$$

where β is a constant.

(ii) For what values of β does $\mathbf{A}^{-1}$ exist?

(iii) Determine $\mathbf{A}^{-1}$ when $\beta = 3$, and verify your answer.

ANSWER

(i)

$$\begin{aligned} \det A &= \det \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ \beta & 5 & 1 \end{pmatrix} \\ &= 1(3(1) - (-1)5) - 2(2(1) - (-1)\beta) + 0 \\ &= 3 + 5 - 2(2 + \beta) \\ &= 8 - 4 - 2\beta = 4 - 2\beta \end{aligned}$$

(ii) A^{-1} exists when $\det A \neq 0$ i.e. $\beta \neq 2$.

(iii) When $\beta = 3,\ A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 3 & 5 & 1 \end{pmatrix}$

By Gaussian elimination

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{array}\right) \quad (r_2' = r_2 - 2r_1, r_3' = r_3 - 3r_1)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 1 \end{array}\right) \quad (r_2' = -r_2)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right) \quad (r_1' = r_1 - 2r_2,\ r_3' = r_3 + r_2)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \quad (r_3' = \tfrac{1}{2} r_3)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 2\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \quad (r_1' = r_1 + 2r_3,\ r_2' = r_2 - r_3)$$

Therefore

$$A^{-1} = \begin{pmatrix} -4 & 1 & 1 \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} -4 & 1 & 1 \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -4+5+0 & 1-1 & 1-1+0 \\ -8+\frac{15}{2}+\frac{1}{2} & 2-\frac{3}{2}+\frac{1}{2} & 2-\frac{3}{2}-\frac{1}{2} \\ -12+\frac{25}{2}-\frac{1}{2} & 3-\frac{5}{2}-\frac{1}{2} & 3-\frac{5}{2}+\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^{-1}A &= \begin{pmatrix} -4 & 1 & 1 \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 3 & 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4+2+3 & -8+3+5 & -1+1 \\ \frac{5}{2}-1-\frac{3}{2} & 5-\frac{3}{2}-\frac{5}{2} & \frac{1}{2}-\frac{1}{2} \\ -\frac{1}{2}-1+\frac{3}{2} & -1-\frac{3}{2}+\frac{5}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$