QUESTION

(a) **(i)** Find the indefinite integral $\int \frac{dx}{4+x^2}$.

(ii) Using partial fractions and the result in (i), if appropriate, find

$$\int \frac{3}{(1+x^2)(4+x^2)}\,dx.$$

(b) Differentiate with respect to x the functions

(i)$\frac{x}{\sqrt{1+x^2}}$, (ii)$\exp(x^2 \sinh x)$.

ANSWER

(a) **(i)** $I = \int \frac{dx}{4+x^2} = \int \frac{dx}{4(1+\frac{x^2}{4})}$ Put $u = \frac{x}{2}$, $\frac{du}{dx} = \frac{1}{2}$ therefore

$I = \int \frac{2du}{4(1+u^2)} = \frac{1}{2}\tan^{-1} u + c = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

(ii)

$$\begin{aligned}
\frac{3}{(4+x^2)(1+x^2)} &= \frac{Ax+B}{4+x^2} + \frac{Cx+D}{1+x^2} \\
&= \frac{(Ax+B)(1+x^2)+(Cx+D)(4+x^2)}{(4+x^2)(1+x^2)}
\end{aligned}$$

i.e. $3 = (Ax+B)(1+x^2)+(Cx+D)(4+x^2)$

Coefficients of:

x^3	A+C=0	(1)
x	A+4C=0	(2)
x^2	B+D=0	(3)
const.	B+4D=3	(4)

$(2)-(1) \Rightarrow 3C = 0 \Rightarrow C = 0 \Rightarrow A = 0$

$(4)-(3) \Rightarrow 3D = 3 \Rightarrow D = 1 \Rightarrow B = -D = -1$

Therefore $\frac{3}{(4+x^2)(1+x^2)} = \frac{1}{1+x^2} - \frac{1}{4+x^2}$

Hence

$$\begin{aligned}
\int \frac{3dx}{(4+x^2)(1+x^2)} &= \int \frac{1}{1+x^2}\,dx - \int \frac{1}{4+x^2}\,dx \\
&= \tan^{-1} x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c
\end{aligned}$$

(b) **(i)**

$$\begin{aligned}\frac{d}{dx}\left\{\frac{x}{\sqrt{1+x^2}}\right\} &= \frac{\sqrt{1+x^2}.1 - x\{\frac{1}{2}(1+x^2)^{-\frac{1}{2}}.2x\}}{1+x^2}\\ &= \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2}\\ &= \frac{(1+x^2)-x^2}{(1+x^2)^{\frac{3}{2}}}\\ &= \frac{1}{(1+x^2)^{\frac{3}{2}}}\end{aligned}$$

(ii)

$$\frac{d}{dx}\{e^{x^2\sinh x}\} = e^{x^2\sinh x}\{x^2\cosh x + 2x\sinh x\}$$