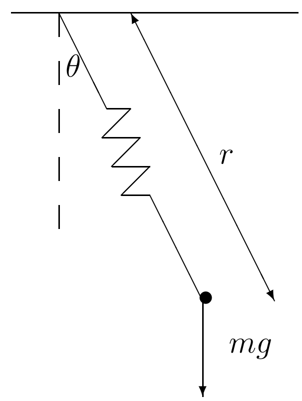
Question

A spring pendulum consists of a mass m on the end of a light spring of stiffness k,the other end of which is fixed to a stationary support. The natural length of the spring is l. Find the Langrangain of the system in terms of r and θ, and hence derive the equations of motion. Putting $r = l + \frac{mg}{k} + \epsilon$ and neglecting all terms of second order of smallness in ϵ and θ show that the equations of motion reduce to

$$m\ddot{\epsilon} + k\epsilon = 0, \qquad \left(l + \frac{mg}{k}\right)\ddot{\theta} + g\theta = 0.$$

Deduce that ω_ϵ, the frequency of radial oscillations, must be greater than ω_θ, the frequency of angular oscillations and that $\omega_\epsilon = 2\omega_\theta$ when $k = \frac{3mg}{l}$.

Answer



$$\begin{aligned}
K.E. &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\
P.E. &= -mgr\cos\theta + \frac{1}{2}k(r-l)^2 \\
L &= \frac{1}{2}m((\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}k(r-l)^2 \\
\frac{\partial L}{\partial \theta} &= -mgr\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 + mg\cos\theta \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}
\end{aligned}$$

Euler-Lagrange equations:

$$\theta: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\sin\theta = 0$$

$$r: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{m}(r-l) = 0$$

Put $r = l + \dfrac{mg}{k} + \epsilon \Rightarrow \dot{r} = \dot{\epsilon}, \ddot{r} = \ddot{\epsilon}$ in the Euler-Lagrange equations and neglect quadratic terms in $\theta, \dot{\theta}, \ddot{\theta}, \epsilon, \dot{\epsilon}, \ddot{\epsilon}$, giving

$$\left(l + \frac{mg}{k}\right)\ddot{\theta} + 0 + g\theta = 0 \Rightarrow \left(l + \frac{mg}{k}\right)\ddot{\theta} + g\theta = 0$$

$$\ddot{\epsilon} - 0 - g + \frac{k}{m}\epsilon = 0 \Rightarrow \ddot{\epsilon} + \frac{k}{m}\epsilon = 0$$

Whence both θ and ϵ undergo simple harmonic motion with frequencies

$$\omega_\theta^2 = \frac{g}{(l + \frac{mg}{k})}, \quad \omega_\epsilon^2 = \frac{k}{m}$$

Therefore $\dfrac{\omega_\epsilon^2}{\omega_\theta^2} = 1 + \dfrac{kl}{mg} > 1 \Rightarrow \omega_\epsilon > \omega_\theta$

Put $k = \dfrac{3mg}{l} \Rightarrow \dfrac{\omega_\epsilon^2}{\omega_\theta^2} = 4 \Rightarrow \omega_\epsilon = 2\omega_\theta$