Question

A particle of mass m slides under gravity on a smooth wire in the shape of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \sin\theta)$, $0 \leq \theta \leq 2\pi$ as shown below PICTURE

(a) Show that the kinetic energy of the bead is $ma^2(1 - \cos\theta)\dot{\theta}^2$.

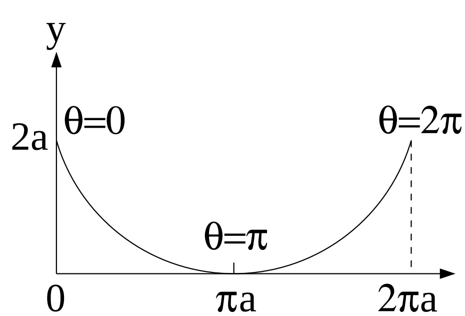
(b) Find the Lagrangian and deduce the equation of motion

$$2\sin\frac{\theta}{2}\ddot{\theta} + \cos\frac{\theta}{2}\dot{\theta}^2 - \frac{g}{a}\cos\frac{\theta}{2} = 0.$$

(c) Rewrite the above equation in terms of $u = \cos\frac{\theta}{2}$ and deduce that, irrespective of the starting position, the period of oscillation is the same as that of a plane pendulum of length $4a$ undergoing small oscillations.

(d) Suppose two identical beads are released from rest with $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. Where do they collide? At what time do they collide? What happens subsequently? (Assume that the coefficient of restitution, e, is unity.)

Answer



$x = a(\theta - \sin\theta) \quad y = a(1 + \cos\theta)$
$\dot{x} = a\dot{\theta}(1 - \cos\theta) \quad \dot{y} = -a\dot{\theta}\sin\theta$

$$\begin{aligned} K.E. &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}ma^2\dot{\theta}^2[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta] \\ &= ma^2\dot{\theta}^2(1 - \cos\theta) \end{aligned}$$

$$\begin{aligned} P.E. &= mgy \\ &= mga(1+\cos\theta) \end{aligned}$$

$$\begin{aligned} L &= K.E. - P.E. \\ &= ma^2\dot{\theta}^2(1-\cos\theta) - mga(1+\cos\theta) \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= 2ma^2(1-\cos\theta)\dot{\theta} \\ &= 4ma^2\dot{\theta}\sin^2\frac{\theta}{2} \end{aligned}$$

$$\frac{\partial L}{\partial \theta} = ma^2\dot{\theta}^2\sin\theta + mga\sin\theta$$

Euler-Lagrange equation:

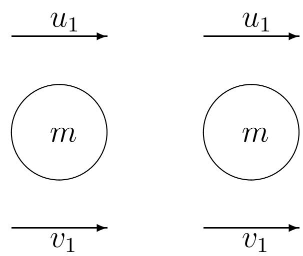
$$\begin{aligned} \frac{d}{dt}\left(4ma^2\dot{\theta}\sin^2\frac{\theta}{2}\right) - ma^2\dot{\theta}^2\sin\theta - mga\sin\theta &= 0 \\ 4ma^2\left[\ddot{\theta}\sin^2\frac{\theta}{2} + \dot{\theta}^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] - ma(a\dot{\theta}^2+g).2\sin\frac{\theta}{2}\cos\frac{\theta}{2} &= 0 \\ 2\sin\frac{\theta}{2} + (\cos\frac{\theta}{2})\dot{\theta}^2 - \frac{g}{a}\cos\frac{\theta}{2} &= 0 \quad (*) \end{aligned}$$

(cancelling $a\sin\frac{\theta}{2} \neq 0$ for general motion.)
$u = \cos\frac{\theta}{2} \quad \Rightarrow \quad \dot{u} = -\frac{1}{2}\sin\frac{\theta}{2}\dot{\theta} \quad \Rightarrow \quad \ddot{u} = -\frac{1}{2}\left[\frac{1}{2}\cos\frac{\theta}{2}\dot{\theta}^2 + \sin\frac{\theta}{2}\ddot{\theta}\right]$
Hence (*) can be written as:

$$\ddot{u} + \frac{g}{4a}u = 0,$$

which implies Simple Harmonic Motion with frequency $\sqrt{\frac{g}{4a}}$ i.e. a pendulum with length 4a.
The period is independent of the starting position so the beads will collide at the bottom of the cardioid $(\pi a, 0)$ after a quarter of a period $\left(2\pi\frac{\sqrt{\frac{4a}{g}}}{4} = \pi\sqrt{\frac{a}{g}}\right)$.
The collisions are perfectly elastic at the bottom.

Before:



After:

$$\text{linear momentum:} \quad \left.\begin{array}{rcl} mu_1 + mu_2 & = & mv_1 + mv_2 \\ v_2 - v_1 & = & -(u_2 - u_1) \end{array}\right\} \Rightarrow \begin{array}{rcl} v_2 & = & u_1 \\ v_1 & = & u_2 \end{array}$$

i.e. the beads exchange their speeds and then repeat the motion (and collisions at the bottom) indefinitely.