Question

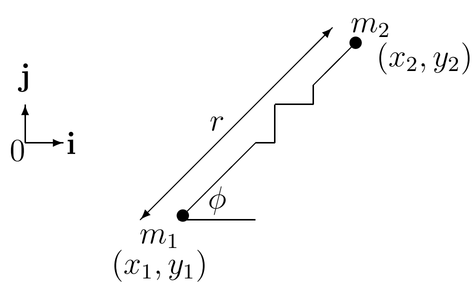
A massless spring of natural length b and spring constant k connects two particles of masses m_1 and m_2. The system rests on a smooth table and may oscillate and rotate. Find the Lagrangian of the system in terms of (x_1, y_1), the Cartesian coordinates of m_1 and (r, ϕ) the polar coordinates of the relative position of m_2 to m_1. Show that

(a) $m_1\dot{x}_1 + m_2\dot{x}_2 =$ constant and $m_1\dot{y}_1 + m_2\dot{y}_2 =$ constant.

(b) $\ddot{r} - r\dot{\phi}^2 + \ddot{x}_1\cos\phi + \ddot{y}\sin\phi + \dfrac{k}{m_2}(l-b) = 0.$

(c) Find the equation of motion for ϕ.

Answer



$x_2 = x_1 + r\cos\phi$
$y_2 = y_1 + r\sin\phi$
$\dot{x}_2 = \dot{x}_1 + \dot{r}\cos\phi - r\dot{\phi}\sin\phi \qquad (*)$
$\dot{y}_2 = \dot{y}_1 + \dot{r}\sin\phi + r\dot{\phi}\cos\phi$

$$\begin{aligned}
L &= K.E. - P.E. \\
&= \frac{1}{2}(m_1+m_2)(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) - \frac{k}{2}(r-b)^2 \\
&= \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{r}^2+r^2\dot{\phi}^2) + m_2\dot{r}(\dot{x}_1\cos\phi + \dot{y}_1\sin\phi) \\
&\quad + m_2 r\dot{\phi}(\dot{y}_1\cos\phi - \dot{x}_1\sin\phi) - \frac{1}{2}k(r-b)^2
\end{aligned}$$

Equations of motion

(a) x_1:

$$\frac{d}{dt}[(m_1+m_2)\dot{x}_1 + m_2\dot{r}\cos\phi - m_2 r\dot{\phi}\sin\phi] = 0$$

$\Rightarrow m_1\dot{x}_1 + m_2\dot{x}_2 = \text{constant}$, (using(*))

y_2:

y_2 : Similarly gives $m_1\dot{y}_1 + m_2\dot{y}_2 = \text{constant}$, (using(*))

(b) r :

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ gives the required answer after some algebra.

(c) ϕ:

$$\begin{aligned} 0 &= \frac{d}{dt}(m_2 r^2\dot{\phi} + m_2 r(\dot{y}_1\cos\phi - \dot{x}_1\sin\phi)) \\ &\quad -(m_2\dot{r}(-\sin\phi\dot{x}_1 + \cos\phi\dot{y}_1) + m_2 r\dot{\phi}(-\dot{y}_1\sin\phi - \dot{x}_1\cos\phi)) \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= 2m_2\dot{\phi}\dot{r}r + m_2 r(\ddot{y}_1\cos\phi - \ddot{x}_1\sin\phi) + m_2 r^2\ddot{\phi} + \\ &\quad m_2\dot{r}(\dot{y}_1\cos\phi - \dot{x}_1\sin\phi) + m_2 r\dot{\phi}(-\dot{y}_1\sin\phi - \dot{x}_1\cos\phi) + \\ &\quad m_2\dot{r}(\dot{x}_1\sin\phi - \dot{y}_1\cos\phi) + m_2 r\dot{\phi}(\dot{y}_1\sin\phi + \dot{x}_1\cos\phi) \end{aligned}$$

Therefore

$$\ddot{\phi} + 2\frac{\dot{\phi}\dot{r}}{r} + \frac{\cos\phi}{r}\ddot{y}_1 - \frac{\sin\phi}{r}\ddot{x}_1 = 0$$