Question

A particle moves in a plane under the action of the force with potential $U(r)$. Write the Lagrangain in terms of the polar coordinates (r, ϕ) and derive the equations of motion of the particle.

Answer

$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi$ Therefore $K.E. = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$

The Lagrangian $L = K.E. - P.E. = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$

Equations of motion:

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - U'; \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r}; \quad \frac{\partial L}{\partial \phi} = 0; \quad \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}.$$

Therefore

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}}\right) - \dfrac{\partial L}{\partial r} = 0 \Rightarrow m[\ddot{r} - r\dot{\phi}^2] = -U'$

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = 0 \Rightarrow \dfrac{d}{dt}(mr^2\dot{\phi}) = 0 \Rightarrow r^2\dot{\phi} = \text{constant}$

Note that these are just the radial and tangential components of Newton's 2nd law in polar coordinates.