Question

Suppose that $X_1, \ldots, X_n$ is a random sample from a Poisson distribution with parameter λ. Using the mgf approach show that the sample sum $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with parameter $n\lambda$.

Answer

Let $X \sim \text{Poisson}(\lambda)$ then mgf of X is $M_X(t) = e^{\lambda(e^t-1)}, \quad -\infty < t < \infty$

Here $Y = \sum_{i=1}^{n} X_i$

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) \\
&= E\{e^{t(X_1+\ldots X_n)}\} \\
&= E\left\{e^{tX_1}.e^{tX_2}\ldots e^{tX_n}\right\} \\
&= E\left(e^{tX_1}\right).E\left(e^{tX_2}\right)\ldots E\left(e^{tX_n}\right) \quad \text{(independence)} \\
&= \left\{E(e^{tX_1})\right\}^n \\
&= \left\{e^{\lambda(e^t-1)}\right\}^n \\
&= e^{n\lambda(e^t-1)}
\end{aligned}$$

But this is the mgf of Poisson with parameter $n\lambda$.

Therefore $Y \sim \text{Poisson}(n\lambda)$ by using the uniqueness theorem of mgf.