Question

Let Y_1 =min$(X_1, \ldots, X_n)$ where $X_1, \ldots, X_n$ is a random sample of size n from the distribution with pdf

$$f(x|\theta) = \frac{1}{\theta}\exp[-(x-\theta)], \quad x > \theta.$$

Show that $2n(Y_1 - \theta)$ has the χ^2 distribution with 2 degrees of freedom.

Answer

We have $f(x|\theta) = e^{-(x-\theta)}, \quad x > \theta$

Therefore

$$\begin{aligned} F(x) &= \int_\theta^x e^{-(u-\theta)}\, du \\ &= \int_0^{x-\theta} e^{-z}\, dz, \quad z = u - \theta \\ &= 1 - e^{-(x-\theta)}, \quad x > \theta \end{aligned}$$

$$\begin{aligned} \text{pdf of } Y_1 &= g(y_1) \\ &= n\{1 - F(y_1)\}^{n-1} f(y_1) \\ &= n\left\{e^{-(y_1-\theta)}\right\}^{n-1} e^{-(y_1-\theta)}, \quad y_1 > \theta \\ &= ne^{-n(y_1-\theta)}, \quad y_1 > \theta \end{aligned}$$

Let $z = 2n(Y_1 - \theta) \Rightarrow Y_1 = \frac{Z}{2n} + \theta, \quad Z > 0$

Therefore $\dfrac{dy_1}{dz} = \dfrac{1}{2n}$. Therefore $\left|\dfrac{dy_1}{dz}\right| = \dfrac{1}{2n}$

Therefore pdf of Z is $h(z) = ne^{-\frac{z}{2}} \cdot \frac{1}{2n} = \frac{1}{2}e^{-\frac{z}{2}}, \quad z > 0$

This is the pdf of χ^2 with 2 degrees of freedom.