Question

Let $X_1, \ldots, X_n$ be a random sample of size n from the distribution having pdf

$$f(x|\theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right), \quad \theta > 0.$$

Let Y_1 =min$(X_1, \ldots, X_n)$ and Y_n =max$(X_1, \ldots, X_n)$. Find the probability density functions of Y_1 and Y_n. Find $E(Y_1)$. Find $P(Y_n \leq 4)$ when $n = 5$.

Answer

We have $f(x|\theta) = \dfrac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty$

Therefore $F(x) = \displaystyle\int_0^x \frac{1}{\theta} e^{-\frac{u}{\theta}}\, du = 1 - e^{-\frac{x}{\theta}}, \quad 0 < x < \infty$

[using: $\displaystyle\int e^{mu}\, du = \frac{e^{mu}}{m} + const$]

pdf of Y_1 is

$$\begin{aligned}
g(y_1) &= n\{1 - F(y_1)\}^{n-1} f(y_1), \quad 0 < y_1 < \infty \\
&= n\left\{1 - e^{-\frac{y_1}{\theta}}\right\}^{n-1} \frac{1}{\theta} e^{-\frac{y_1}{\theta}}, \quad 0 < y_1 < \infty \\
&= \frac{n}{\theta} e^{-\frac{(n-1)y_1}{\theta}} \cdot e^{-\frac{y_1}{\theta}}, \quad 0 < y_1 < \infty \\
&= \frac{n}{\theta} e^{-\frac{n}{\theta} y_1}, \quad 0 < y_1 < \infty
\end{aligned}$$

We see that $Y_1 \sim$ Exponential with $\beta = \frac{\theta}{n}$

[Table of common distributions]

Therefore $E(Y_1) = \beta = \frac{\theta}{n}$

pdf of Y_n is

$$\begin{aligned}
h(y_n) &= n\{F(y_n\}^{n-1} f(y_n) \\
&= n\left\{1 - e^{-\frac{y_n}{\theta}}\right\} \frac{1}{\theta} e^{-\frac{y_n}{\theta}} \\
&= \frac{n}{\theta} e^{-\frac{y_n}{\theta}} \left(1 - e^{-\frac{y_n}{\theta}}\right)^{n-1}, \quad 0 < y_n < \infty
\end{aligned}$$

$P(Y_n \leq y_n) = \{F(y_n)\}^n$

Therefore $P(Y_n \leq 4) = \{F(4)\}^n = \left(1 - e^{-\frac{4}{\theta}}\right)^n$ for any n

Therefore Answer $= \left(1 - e^{-\frac{4}{\theta}}\right)^5$ when $n = 5$