Question

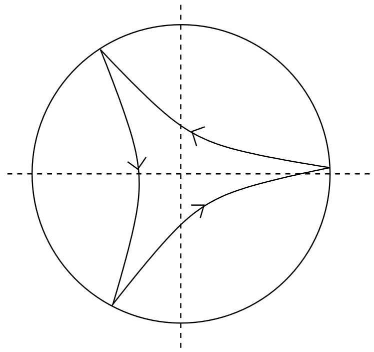
In each case sketch the curve γ and find its total path length:

(i) $\gamma(t) = (2\cos t + \cos 2t, 2\sin t - \sin 2t)$, $0 \le t \le 2\pi$;

(ii) $\gamma(t) = (\sin^3 t, \cos^3 t)$, $0 \le t \le 2\pi$.

Answer

(i) Hypocycloid: point of radius 1 rolling inside a circle of radius 3.



$$\begin{aligned}
\gamma'(t) &= (-2\sin t - 2\sin 2t, 2\cos t - 2\cos 2t)\\
\|\gamma'(t)\|^2 &= 4(1 + \sin t \sin 2t - 2\cos t\cos 2t + 1)\\
&= 8(1 - \cos 3t)\\
\Rightarrow \textbf{length} &= \int_0^{2\pi} \sqrt{8}(1 - \cos 3t)^{\frac{1}{2}}\,dt\\
&= \sqrt{8}\sqrt{2}\int_0^{2\pi} \underbrace{\left|\sin\frac{3t}{2}\right|}_{note!}\,dt\\
&= 4 \times 3\int_0^{\frac{2\pi}{3}} \sin\frac{3t}{2}\,dt = 16
\end{aligned}$$

(ii)

$$\begin{aligned}
x &= s^3\\
y &= c^3\\
\|\gamma'(t)\| &= \|(3s^2c, -3c^2s)\|\\
&= 3|sc|\\
\Rightarrow \textbf{length} &= 4\int_0^{2\pi} \frac{3}{2}\sin 2t\,dt\\
&= 4\left[-\frac{3}{4}\cos 2t\right]_0^{2\pi} = 6
\end{aligned}$$