Question

($*$) To help determine a possible strategy for future whaling the population is modelled mathematically.

1. In the absence of any fishing occurring the population of humpback whales, $Y(t)$, can be approximated as obeying the logistic equation

$$\frac{dY}{dt} = rY(1 - Y/K) \; .$$

   If $Y(0) = K/3$ find $Y(t)$ and hence the time τ at which the population has doubled.

2. The model can be extended to take "harvesting" of the whales into account. A simple model is to assume that the rate at which whales are caught, called the yield, is proportional to the population of whales. Specifically the yield is taken to be EY (where E is a constant determined by the amount resources devoted to catching the whales) and the model is then

$$\frac{dY}{dt} = rY(1 - Y/K) - EY$$

   (This is known as the *Schaefer model.*)
   Show that if $E < r$ there are two equilibrium points $Y = 0$ and $Y = K(1-E/r)$ and that the first of these is unstable and the second stable. From this solution find the yield (ie: $EY =$ the rate at which whales will be caught) that will occur after a long time (this is called the *sustainable yield*). Find the value of E which gives the maximum sustainable yield (and hence the level of whaling that will result in the maximum number of whales being caught on a sustainable basis).
   Comment on what might occur if we take $E > r$?

Answer

a) $\dfrac{dY}{dt} = rY\left(1 - \dfrac{Y}{k}\right) \qquad Y(0) = \dfrac{k}{3}$

solve by separation of variables $\displaystyle\int \frac{dY}{Y\left(1 - \frac{Y}{k}\right)} = \int r dt$

Now by partial fractions $\int \frac{1}{Y} + \frac{\frac{1}{k}}{1 - \frac{Y}{k}} dY = rt + A$

$\ln Y - \ln(k - Y) = rt + A$

use the initial data $\ln Y - \ln(k - Y) = rt + \ln\left(\frac{k}{3}\right) - \ln\left(\frac{2k}{3}\right)$

find τ where $Y(\tau) = \frac{2k}{3}$

$\Rightarrow \ln \frac{2k}{3} - \ln\left(k - \frac{2k}{3}\right) = r\tau + \ln\left(\frac{k}{3}\right) - \ln\left(\frac{2k}{3}\right)$

$\Rightarrow \ln 2 = r\tau - \ln 2 \Rightarrow \tau = \frac{2}{r} \ln 2$

b) plot solution curves of $\frac{dY}{dt} = rY\left(1 - \frac{Y}{k}\right) - EY$

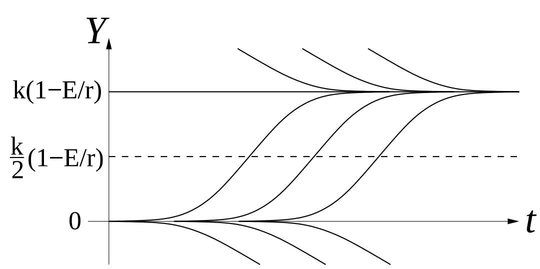
isoclines $rY\left(1 - \frac{Y}{k}\right) - EY = c,$

$c = 0 \Rightarrow Y = 0$ and $\mathrm{Y} = \mathrm{k}\left(1 - \frac{\mathrm{E}}{\mathrm{r}}\right)$

c is maximum when $Y = \frac{k}{2}\left(1 - \frac{E}{r}\right)$

$c > 0, \quad 0 < Y < k\left(1 - \frac{E}{r}\right)$

$c < 0 \begin{cases} Y < 0 \\ Y > k\left(1 - \frac{E}{r}\right) \end{cases}$



Solution curves for $1 - \frac{E}{r} > C$

Equilibrium at $Y = 0$ which is unstable, and at $Y = k\left(1 - \frac{E}{r}\right)$ which is stable.

The yield is EY so at $t \to \infty$, $\;Y \to k\left(1 - \frac{E}{r}\right)$

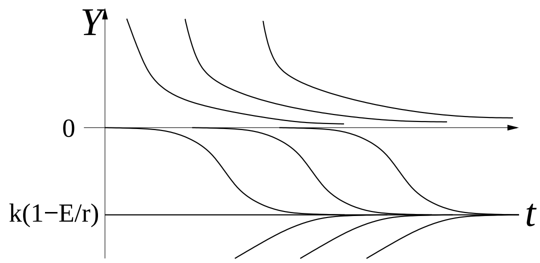
therefore yield $\to Ek\left(1 - \frac{E}{r}\right)$.

The maximum yield occurs when $\dfrac{d(YE)}{dE} = 0$

$\Rightarrow k\left(1 - \frac{2E}{r}\right) = 0 \Rightarrow E = \frac{r}{2}$

i.e. catch fish at half the rate they are born.

If $E > r$ then the solution curves are:



Hence for physically relevant use of $Y \geq 0$ we have only one equilibrium ($Y = 0$) and it is stable. If we harvest faster than the fish are born ($EY > rY$) then we slowly lose the population of fish.